Solutions to Homework Set 3

February 25, 2000

1 Problem 1

1.1 a) Equation for $f(x',y')$

$f(x',y')$ is merely a sinusoidal function with period 16 along the $x'$ axis. Thus:

$$f(x', y') = \sin \left( \frac{2\pi x'}{16} \right)$$  \hspace{1cm} (1)

1.2 b) Equation for $f(x,y)$

To write $f(x,y)$, we must rotate the coordinate system $x'y'$. In other words, we write $x'$ and $y'$ as a function of $x$ and $y$, and substitute them on the original expression for $f(x',y')$. The coordinate transformation equations are:

$$x' = x\cdot \cos(30^\circ) + y\cdot \sin(30^\circ)$$

$$y' = -x\cdot \sin(30^\circ) + y\cdot \cos(30^\circ),$$

Since $f(x',y')$ is independent of $y'$, we only use the top equation:

$$x' = 0.87x + 0.5y$$

Substituting $x'$ in equation 1 gives:

$$f(x, y) = \sin \left( \frac{2\pi (0.87x + 0.5y)}{16} \right)$$  \hspace{1cm} (2)

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1.3  c) Profiles of f(x’,0), f(x,0), and f(y,0)

See Figure 2

2  Problem 2

Checkerboard pattern is made from a 2D rectangular function centered at the origin, and replicated in both X and Y direction. The replication uses two sets of delta function patterns, as shown in figure 1. One is shifted by 8 in the x and y directions(squares), and the other is centered at the origin (dots). Both have the period equals 16.

![Checkerboard Pattern](image)

Figure 1: Checkerboard functions. (1) \( \text{rect}(\frac{x}{8}, \frac{y}{8}) \) (center). (2) \( \delta(x - 8, y - 8) \) (squares). (3) \( \delta(x - 16, y - 16) \) (dots)

2.1  Equation 1

We define the center rectangle as \( \text{rect}(\frac{x}{8}, \frac{y}{8}) \), and use the function
\[ (\sum_{n=-\infty}^{+\infty} \delta(x - 8 - n16, y - 8 - n16)) + (\sum_{n=-\infty}^{+\infty} \delta(x - n8, y - n8)), \]
to replicate the rectangle. We then write \( f(x, y) \) as:

\[ f(x, y) = \text{rect}(\frac{x}{8}, \frac{y}{8}) \ast \left[ (\sum_{n=-\infty}^{+\infty} \delta(x - 8 - n16, y - 8 - n16)) + (\sum_{n=-\infty}^{+\infty} \delta(x - n16, y - n16)) \right] \quad (3) \]

2.2 Equation 2

We can write the delta functions as comb functions:

\[ (\sum_{n=-\infty}^{+\infty} \delta(x - 8 - n16, y - 8 - n16)) = \frac{1}{(16)^{\frac{1}{2}}} \cdot \text{comb}(\frac{x-8}{16}, \frac{y-8}{16}) \]

and

\[ (\sum_{n=-\infty}^{+\infty} \delta(x - n16, y - n16)) = \frac{1}{(16)^{\frac{1}{2}}} \cdot \text{comb}(\frac{x}{16}, \frac{y}{16}) \]

Then,

\[ f(x, y) = [\text{rect}(\frac{x}{8}, \frac{y}{8})] \ast \left[ \left( \frac{1}{(16)^{\frac{1}{2}}} \right) \cdot \text{comb}(\frac{x-8}{16}, \frac{y-8}{16}) + \text{comb}(\frac{x}{16}, \frac{y}{16}) \right] \quad (4) \]

3 Problem 3

3.1 Fourier Transform of Function in Figure A

The Fourier Transform of function 2 can be found using equation 1, and then rotating the coordinating systems in the spatial frequency domain. Using the table transform pair:

\[ \sin(2\pi \eta_0 x) = \frac{j}{2\eta_0 x} \delta(u) \delta(\frac{\eta}{\eta_0}) \]

we get,

\[ FT[f(x', y')] = FT[\sin(\frac{2\pi x'}{16})] = \frac{j}{2\eta_0 x'} \delta(u') \delta(\frac{\eta'}{\eta_0}) \]

In order to rotate the coordinate system, we use the following equations for \( u' \), and \( v' \):

\[ \begin{align*}
\end{align*} \]
\[ u' = 0.87u + 0.5v \text{ and } v' = -0.5u + 0.87v \]

Substituting \( v' \) and \( u' \) in the FT equation and after some simplification:

\[ F_1(u, v) = 8j\delta(0.87u + 0.5v)\delta(16(-0.5u + 0.87v)) \] (5)

The plot of the magnitude of \( F_1 \) is in Figure 3.

### 3.2 Fourier Transform of Function in Figure B

The Fourier Transform of function 4 is given by:

\[ F_2(u, v) = FT[rect\left(\frac{x}{8}, \frac{y}{8}\right)]FT[\frac{1}{16}\delta\cdot[comb\left(\frac{x-8}{16}, \frac{y-8}{16}\right) + comb\left(\frac{x}{16}, \frac{y}{16}\right)]] \]

Which from the table of 2-D transform pairs, translates to:

\[ F_2(u, v) = (8)^2.sinc(8u, 8v)\cdot\left(\frac{16}{16}\delta[comb(16u, 16v) + comb(16u, 16v).e^{-j2\pi(8u+v)}] \right) \]

The exponential term in the above equation is due to the shift on the second comb of equation 4. Simplifying the expression results in:

\[ F_2(u, v) = 64sinc(8u, 8v).[comb(16u, 16v) + comb(16u, 16v).e^{-j2\pi(8u+v)}] \] (6)

The plot of the magnitude of \( F_2 \) is in Figure 4.

### 4 Problem 4

The checkerboard image can be constructed using several variations of checkerboard images, with different aspect ratios, or squarewise constructions of uniform images. Then the final image can be put together using the 'insert' command on the 'Geometry' menu.

Following the procedure described in the assignment, the Fourier Transform of the image is computed using the 'power spectrum' function from the 'filter' menu, and the 'square root stretch' from the 'contrast' menu. Multiple stretches can be processed on the image to enhance its features. The result is shown in figures 5 (amplitude) and 6. Compare the amplitude with image in figure 4.
Figure 2: Plots of $f(x',0)$, $f(x,0)$, and $f(y,0)$ profiles
Figure 3: Top View Image of F1(u,v)
Figure 4: Top View Image of $F_2(u,v)$
Figure 5: Checkerboard Power Spectrum
Figure 6: Checkerboard Phase