1) Given the following integration algorithm described by the Butcher tableau:

\[
\begin{array}{c|cccccc}
& 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 \\
x & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\end{array}
\]

a) Write down the stages of this algorithm.

b) Is the algorithm explicit or implicit?
2) For the algorithm of problem #1, find the linear order of approximation accuracy by plugging in the test problem:
\[ \dot{x} = A \cdot x \]

The algorithm is described in the literature as Runge-Kutta-Simpson algorithm.
3) In the early 70's, the most widely used simulation software was a program from IBM, called CSMP-III. It offered as its default integration algorithm the standard 4th-order RK algorithm introduced in Chapter 3 of the book. For step-size control, it used a second algorithm, which was a variant of Simpson's rule:

\[ x_{k+1} = x_{k-1} + \frac{h}{3} (f_{k+1} + 4f_k + f_{k-1}) \]
Simpson's rule, also known under the name 4th-order Milne algorithm, is an implicit multistep algorithm. However, this was not how the algorithm was implemented in CSMP-III. We can rewrite the algorithm as:

\[ x_{k+1} = x_k + \frac{h}{6} (-f_{k+1} + 4f_{k+\frac{1}{2}} + f_k) \]

by simply cutting the step size in half. CSMP-III implemented this formula as a predictor/corrector technique.
\[ x_{P_i}^k = x_k + \frac{h}{2} \cdot x_{P_1}^k \]  \hspace{1cm} (FE)

\[ x_{P_1}^k = f \left( x_{P_i}^k, t_{k+\frac{1}{2}} \right) \]

\[ x_{P_2}^k = x_{P_1}^k + \frac{h}{2} \cdot x_{P_1}^k \]  \hspace{1cm} (FE)

\[ x_{P_2}^k = f \left( x_{P_2}^k, t_{k+\frac{h}{2}} \right) \]

\[ x_c = x_k + \frac{h}{6} \left( x_{P_2}^k + 4 \cdot x_{P_1}^k + x_k \right) \]

The predictor/corrector technique is of course explicit, thus it has different stability and accuracy properties than the implicit formula.

a) Write down a Butcher tableau for the combined RK/Simpson method with 4 stages.
as possible.

b) Find $f(q)$ for the Simpson method.

c) What is the linear order of approximation accuracy of the method?

d) How would you classify this method?

4) For $BI 4/5\theta_{.45}$, find $f(q)$, assuming that the two semi-steps are implemented using RKF4/5.
5) Given the Van der Pol oscillator:

\[
\ddot{x} - \mu (1 - x^2) \dot{x} + x = 0
\]

a) Write down a state-space description using the state variables:

\[
\begin{align*}
    x_1 &= x \\
    x_2 &= \dot{x}
\end{align*}
\]

b) We wish to simulate the model using a 3rd-order accurate technique:

\[
x_{k+1} = x_k + h \cdot f_k + \frac{h^2}{2} \cdot f_k + \frac{h^3}{6} \cdot \ddot{f}_k
\]
We already have the equations for $f_k$. That is our model. Add symbolic equations for $f_{\dot{k}}$ and $f_{\ddot{k}}$ to the model using algebraic differentiation.