1) Prove that the Hankel norm of a system is invariant to similarity transformations.

2) The Matlab routine:

\[ [U, T] = \text{schur} \ (A) \]

finds a unitary transformation of \( A \):

\[ A = U \cdot T \cdot U^* \]

such that \( T \) is in block-upper-triangular form.

Write a Matlab routine:

\[ [U, T] = \text{schur2} \ (A) \]

that finds a unitary transformation of \( A \):

\[ A = \overline{U} \cdot T \cdot U^* \]
Such that $T$ is in block-lower-triangular form.

Hint: schur2 may call schur.

3) The optimal control problem:

$$\dot{x} = Ax + Bu$$

$$P_1 = \int_0^\infty (x^TQx + u^TRu) \, dt \overset{!}{=} \text{min.}$$

can be solved by the following matrix-Riccati equation:

$$A^*P + PA - PBR^{-1}B^*P = -Q$$

$$\Rightarrow K = R^{-1}B^*P$$

$$\Rightarrow u_{opt} = v - K \cdot x$$

is the optimal state feedback.
We want to study the effect of similarity transformations on the control problem.

a) Given the similarity transformation:

\[
\mathbf{x}_n = T \cdot \mathbf{x}
\]

Reformulate the same optimal control problem (same PI) in terms of the new state vector \( \mathbf{x}_n \):\[
\mathbf{P}_I = \int (\mathbf{x}_n^* \mathbf{Q} \mathbf{x}_n + \mathbf{u}^* \mathbf{R} \mathbf{u}) \, dt
\]

What is the effect of the similarity transformation on \( \mathbf{Q} \)?

= Find \( \mathbf{Q} \) as function of \( \mathbf{Q} \) and \( T \).
b) Given an optimal control problem with Hermitian, but not diagonal $Q$ matrix.

Show that there exists a unitary transformation:

$$\Sigma = U \cdot X$$

such that $\Sigma$ will be diagonal.

Find $U$.

c) Determine the effect of the similarity transformation:

$$\tilde{\Sigma} = T \cdot X$$

on the $P$-matrix, i.e., find $\tilde{P}$ as a function of $P$ and $T$. 
d) Find a unitary transformation:
\[ X = U \cdot X \]
that makes \( P \) diagonal. Determine \( U \).

e) Determine the effects of the similarity transformation:
\[ Y = \overline{T} \cdot X \]
on \( K \), i.e., find \( K \) as a function of \( K \) and \( T \).

f) Find a unitary transformation:
\[ Z = U \cdot X \]
that makes \( K \) lower triangular. Determine \( U \).