Numerical Computation of Eigenvalues

Given:

\[ \dot{x} = Ax + bu \]
\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} ; \quad b = \begin{bmatrix} \phi \\ \phi \\ \phi \\ \phi \\ -\phi \end{bmatrix} \]

This is:

\[ \rightarrow \text{a Jordan-canonical form with a multiple eigenvalue:} \]
\[ \lambda_1 = 0 ; \quad m_1 = n \]

\[ \rightarrow \text{a controller-canonical form with the characteristic polynomial} \]
\[ \lambda^n = 0 \]

Let us perturb the system with a small \( \varepsilon \):
\[ A = \begin{bmatrix} \lambda^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \\ \end{bmatrix} \quad ; \quad b = \begin{bmatrix} \Theta \\ \vdots \\ \vdots \\ \vdots \\ \Theta \\ \end{bmatrix} \]

This is no longer a Jordan-canonical form. However, it is still a controller-canonical form with the characteristic polynomial:

\[ \lambda^n - \Theta = 0 \]

\[ \Rightarrow \lambda_i = \sqrt[n]{\Theta} \]

Let us write \( \Theta \) as a complex number in polar coordinates:

\[ \Theta = |\Theta| \cdot e^{j2\pi} \]

\[ \Rightarrow \lambda_i = \sqrt[n]{\Theta} = \sqrt[n]{|\Theta|} \cdot e^{j\frac{2k\pi}{n}} \]

\[ \text{\begin{figure}[h]
\begin{center}
\includegraphics[width=0.5\textwidth]{circle.png}
\end{center}
\end{figure}} \]
Let: \[ \varepsilon = 10^{-10} \]
\[ n = 10^6 \]
\[ \Rightarrow \sqrt{\varepsilon} = 0.1 \]

A small change in the parameters of the system matrix \( A \) had a huge effect on the location of the eigenvalues.

This is generally true in the vicinity of multiple poles.

Remember:

\[ \text{Let us draw a root locus for } k \in [0, \infty) \text{ in increments} \]
The sensitivity of the pole location is much bigger in the vicinity of \(-0.5\), where the root locus shows a double pole.

- Once the eigenvalues are known precisely, computation of the corresponding eigenvectors is comparatively benign from a numerical
-5-

perspective.

- Computing the eigenvalue location accurately may not be possible.

- The worst case is the one illustrated before. Let $\sigma_i = \|A\|_2$ be the largest singular value.

$$\Rightarrow \text{err}(\lambda_i) \leq \sigma_i \cdot \varepsilon^{-\frac{1}{2n}}$$

The eigenvalues cannot be computed more accurately than $\sqrt{n} \cdot \sigma_i$ in the worst case.

- We shall need to discuss algorithms to assess whether the matrix $A$ has eigenvalues that are close to being multiple.
Recipe: Given $A$ with
$$\lambda_i = \text{eig}(A).$$

We compute the following perturbation:
$$\hat{A} = A + \varepsilon \cdot \|A\|_2 \cdot \text{rand}(|\text{size}(A)|)$$
$$\Rightarrow \hat{\lambda}_i = \text{eig}(\hat{A})$$

Then:
$$\text{cond} = \frac{\|\lambda_i - \hat{\lambda}_i\|_\infty}{\varepsilon}$$
can be used to estimate the
sensitivity of the pole locations to perturbations of $A$. You may want to
repeat a few times.

Cond $\approx 1 \Rightarrow$ all eigenvalues are distinct

Cond $\gg 1 \Rightarrow$ multiple or almost multiple eigenvalues are present.