A TIME-SCALE METHOD FOR MODEL REDUCTION OF DISCRETE-TIME SYSTEMS

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VIII.7. Minimization of equation error for discrete-time systems

A model reduction method based on the minimization of equation error has been proposed by Eitelberg (1978) for continuous-time linear systems. In the following, the version for discrete-time systems will be developed.

Given the n-th order linear-discrete-time system described by

\[ x(k+1) = Ax(k) + Bu(k) \]
\[ y(k) = cx(k) \]

We assume that the m-th order reduced model is described by

\[ \tilde{x}(k+1) = A_r \tilde{x}(k) + B_r u(k) \]

The states of interest \( x_r \) could be "picked-out" from the original state vector by means of a \((m \times n)\) "masking matrix" \( R \) with the elements 1 and zero such that

\[ x_r = Rx \]

It's now required to represent the m-dimensional state vector \( x_r \) by the m-th order reduced model, i.e. for \( \tilde{x} \approx x_r \), we may write

\[ x_r(k+1) \approx A_r x_r(k) + B_r u(k) \]

We write the equation error as

\[ e(k+1) = x_r(k+1) - A_r x_r(k) - B_r u(k) \]
For \( x(0) = 0 \), \( x_r(0) = 0 \), \( u(k) \equiv \) step function = \[ \begin{cases} u_0, & k \geq 0 \\ 0, & k < 0 \end{cases} \]

the equation error becomes

\[
e(k+1) = Rx(k+1) - A_r Rx(k) - B_r U_0
\]

\[
= (RA_r - A_r R)x(k) + (RB_r - B_r)U_0
\]

(8.22)

But \( x(k) = A^k x(0) + \sum_{i=1}^{k} A^{i-1} B u(k-i) \)

and since, by assumption, \( x(0) = 0 \), \( u(k) = U_0 \) for \( k \geq 0 \), then

\[
x(k) = \sum_{i=1}^{k} A^{i-1} B U_0
\]

(8.23)

substituting from (8.23) into (8.22), we obtain

\[
e(k+1) = (RA_r - A_r R) \sum_{i=1}^{k} A^{i-1} B U_0 + (RB_r - B_r)U_0
\]

(8.24)

In (8.24), \( U_0 \) is just a scaling factor and will be dropped defining

\[
E(k) = (RA_r - A_r R) \sum_{i=1}^{k} A^{i-1} B + (RB_r - B_r)
\]

\[= (RA_r - A_r R) \sum_{j=0}^{k-1} A^j B + RB_r\]

Assuming \( A \) is stable, then the above matrix-series will converge and we may write

\[E(k) = (RA_r - A_r R)(I - A^k)(I - A)^{-1} B + RB_r\]

(8.25)
The stationary value of $x$ is obtained from (8.23) by letting $k$ tends to infinity, i.e.

$$x_{st} = x(\infty) = \lim_{j=0}^{\infty} A^j B U_0 = (I-A)^{-1} B U_0$$

which exists for all stable $A$.

We wish to have

$$x_{r \ st} = R x_{st} \ , \ i.e.$$

$$R(I-A_r)^{-1} B_r U_0 = R(I-A)^{-1} B U_0 \quad (8.26)$$

From (8.25) yields $B_r$ that matches the steady-state response of the original states and those of the reduced model.

$$B_r = (I-A_r)R(I-A)^{-1} B \quad (8.27)$$

Substituting from (8.27) into (8.25), and after few algebraic manipulations, we obtain

$$E(k) = [R-(I-A_r)R(I-A)^{-1}]A^k B \quad (8.28)$$

Now, we wish to determine $A_r$ that minimizes the "error measure"

$$q = \sum_{k=o}^{\infty} \| E(k) \|^2$$

which can be rewritten as

$$q = \sum_{k=o}^{\infty} \text{trace} \{ E(k) E^T(k^l) \} \quad (8.29)$$
But, \( E(k)E^T(k) = [R-(I-A_r)R(I-A)^{-1}]A^kBB^T A^k^T \[R-(I-A_r)^{-1}] \]

then

\[ q = \text{trace} \ P S P^T \] \hspace{1cm} (8.30)

where

\[ P = [R-(I-A_r)R(I-A)^{-1}] \]

\[ S = \sum_{k=0}^{\infty} A^kBB^T A^k^T \]

Differentiating \( q \) after \( A_r \) and letting, \( \frac{dq}{dA_r} = 0 \), we obtain the optimal matrix \( A_r^* \)

\[ A_r^* = I-RSD^TDSD^T \] \hspace{1cm} (8.31)

where

\[ D = R(I-A)^{-1} \]

\( S \) is the solution of the discrete-Lyapunov equation

\[ ASA^T-S = -BB^T \] \hspace{1cm} (8.32)

The reduced model is obtained by first solving equation (8.32) for \( S \) then substituting in (8.31) to obtain \( A_r^* \). Substitution in (8.27) yields the input matrix \( B_r \).