1) For many applications, it is important to be able to compute the inverse of the system matrix. For example:

\[ G(s) = C \cdot (sI-A)^{-1} \cdot B + D \]

\[ \Rightarrow G(\phi) = -C \cdot A^{-1} \cdot B + D \]

Given \( A \) in observer-canonical form:

\[
A = \begin{bmatrix}
\phi & 0 & \cdots & 0 & -a_0 \\
0 & \phi & \cdots & \cdots & -a_1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \phi & -a_{n-1} \\
0 & 0 & \cdots & 0 & \phi
\end{bmatrix}
\]

Determine \( A^{-1} \).

Hint: Decompose \( A \) suitably into four submatrices:
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]

Then look at the solution of the problem: \( y = A \cdot x \)
\[ \Rightarrow x = A^{-1} \cdot y \]

2) A good way to determine the controllability of a system is by computing the controllability gramian, \( G_c \), of the system from the following Lyapunov equation:
\[ A \cdot G_c + G_c \cdot A^* = -B \cdot B^* \]
If \( G_c \) has full rank, then the system is controllable.
a) Determine how $\mathbf{G}_c$ changes under similarity transformation

$$\mathbf{G}_c = \mathbf{T} \cdot \mathbf{A} \cdot \mathbf{G}_c \cdot \mathbf{A}^* = \mathbf{R} \cdot \mathbf{R}^*$$

$$\Rightarrow \mathbf{G}_c = f(\mathbf{G}_c, \mathbf{T})$$

Find $f$.

b) Observability can be determined using the duality principle. Apply this principle to determine the Lyapunov equation that will compute the observability gramian, $\mathbf{G}_o$.

c) Analogous to (a), determine how $\mathbf{G}_o$ changes under similarity transformations.
d) Prove that the product $G_c \cdot G_c$ has the property that its eigenvalues are invariant to similarity transformations.

3) Given the Lyapunov equation that computes the controllability gramian, $G_c$:

$$A \cdot G_c + G_c \cdot A^* = -B \cdot B^*$$

a) Show that there exists a unitary transformation:

$$X = T \cdot X$$

that transforms the above Lyapunov equation to the simpler form:

$$\hat{A} \cdot \hat{G}_c + \hat{G}_c \cdot \hat{A}^* = -\Sigma^2$$
where $\Sigma$ is diagonal and real-valued.

Hint: Apply an SVD to $B$.

b) Matlab offers a routine:

$$G_c = \text{gram}(A,B)$$

that computes the controllability gramian.

How can you use this routine to determine the observability gramian?

c) Develop a Matlab routine:

$$G_c = \text{gram}_2(A,B)$$

that applies the algorithm of (a) to get the right hand
side of the Lyapunov equation
to diagonal form, then
calls upon the gram routine
to compute $G_C$, then undoes
the damage to get $G_C$.

4) Given the polynomial matrix
representation:

$$\begin{bmatrix}
  s^2 & (s+1) & -3 \\
  0 & 2 & 0 \\
  s & s^2 & (s+3)
\end{bmatrix} \cdot \mathbf{Z}(t) = \begin{bmatrix}
  1 & s \\
  s & -1
\end{bmatrix} \cdot \mu(t)$$

$$\mu(t) = \begin{bmatrix}
  s^2 & s^2 & 1 \\
  s^2 & 1 & s
\end{bmatrix} \cdot \mathbf{Z}(t) + \begin{bmatrix}
  0 & 0 \\
  0 & 1
\end{bmatrix} \cdot \mu(t)$$

The 2nd partial state is
unimportant. Rewrite the
polynomial matrix representation
by eliminating the unimportant state.
5) Given the system:

\[ G(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + a_2 s^2 + s^3} + d \]

which can obviously be realized as:

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = \begin{bmatrix}
\phi & 1 & \phi \\
-\phi & 0 & 1 \\
-\phi_0 & -\phi_1 & -\phi_2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
\phi \\
\phi_0 \\
\phi_1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
b_0 & b_1 & b_2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
d
\end{bmatrix} u
\]

a) Write a polynomial matrix representation of this system:

\[
\begin{bmatrix}
\Phi(s) = sI - A \\
Q(s) = B \\
R(s) = C \\
W(s) = D
\end{bmatrix}
\]
b) Multiply the partial state equation from the left with:

\[
U(s) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{s}{(s^2+a_2s+a_1)} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\dot{x}(t) = V(s) \cdot \hat{x}(t)\]

with:

\[
V(s) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
-\frac{1}{s} & \frac{1}{s} & 0 & 0 \\
\frac{1}{s} & 0 & \frac{1}{s} & 0 \\
0 & -\frac{1}{s} & 0 & \frac{1}{s^2}
\end{bmatrix}
\]

which transforms our former system:

\[
\begin{bmatrix}
\hat{P}(s) \cdot \hat{x}(t) = \hat{Q}(s) \cdot \nu(t) \\
y(t) = R(s) \cdot \hat{x}(t) + W(s) \cdot \nu(t)
\end{bmatrix}
\]
to the similar form:

\[
\begin{align*}
\hat{P}(s) \cdot \hat{z}_1(t) &= \hat{Q}(s) \cdot \hat{u}(t) \\
y(t) &= \hat{R}(s) \cdot \hat{z}_2(t) + \hat{W}(s) \cdot \hat{u}(t)
\end{align*}
\]

This system is now in a special canonical form. Which form is this?

d) What can you say about the importance of the first two partial state variables \( \hat{z}_1(t) \) and \( \hat{z}_2(t) \)?

e) What can you say about the controllability of these partial state variables?
f) Extract the controllable and observable subsystem $S_1$ of
the polynomial matrix
representation, assuming that:
\[ p(s) = b_0 + b_1 s + b_2 s^2 \]
\[ q(s) = a_0 + a_1 s + a_2 s^2 + s^3 \]
\[ [p(s), q(s)] \text{ relative prime.} \]

g) Generalize $U(s)$ and $V(s)$
to systems of arbitrary order.
Given any SISO system in
CCF:
\[
\begin{align*}
\dot{x} &= A_{ccf} \cdot x + b_{ccf} \cdot u \\
y &= c_{ccf} \cdot x + d_{ccf} \cdot u
\end{align*}
\]
Find $U(s)$ and $V(s)$ that transform the polynomial matrix representation to the desired canonical form.