The Structural Singularity Removal Algorithm by Pantelides

- This lecture deals with a procedure that can be used to remove structural singularities from a model in a systematic and algorithmic fashion. It is called the ***Algorithm of Pantelides***.
- The algorithm of Pantelides is a symbolic index-reduction algorithm.

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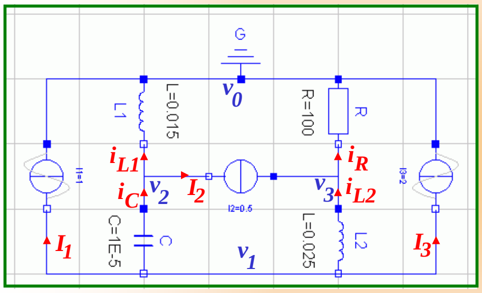
Structural Singularities: An Example I



We compose a model using the currents, the Voltages and the potentials. Consequently, the mesh equations are being ignored.

We have 7 circuit components plus the ground, therefore 2×7 + 1 = 15 equations. In addition, there are 4 nodes giving rise to another 3 equations. Therefore, we expect 18 equations in 18 unknowns.

For passive components, it is customary to normalize the Voltages in the same direction as the currents. For active components (sources), the reverse is true.

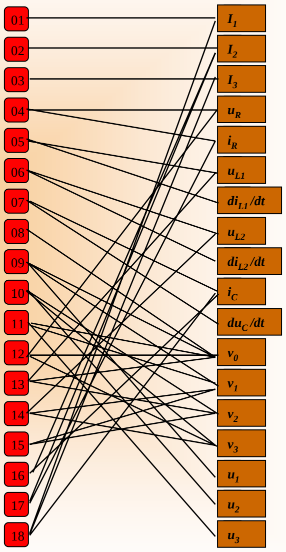
Structural Singularities: An Example II

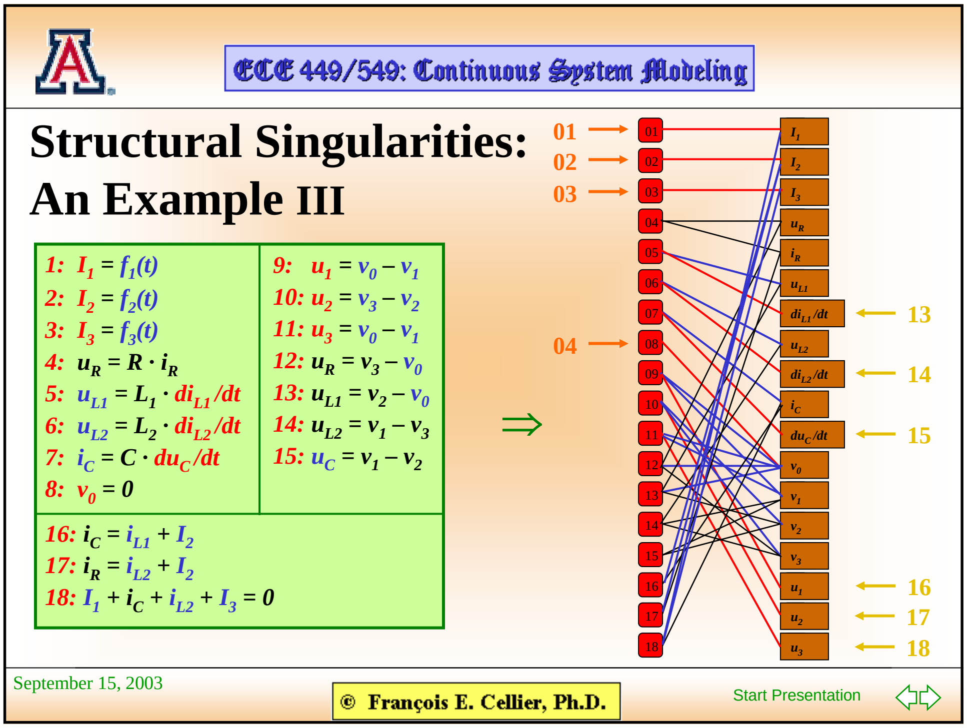
1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$

⇒





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Structural Singularities: An Example III

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$

$\Rightarrow$

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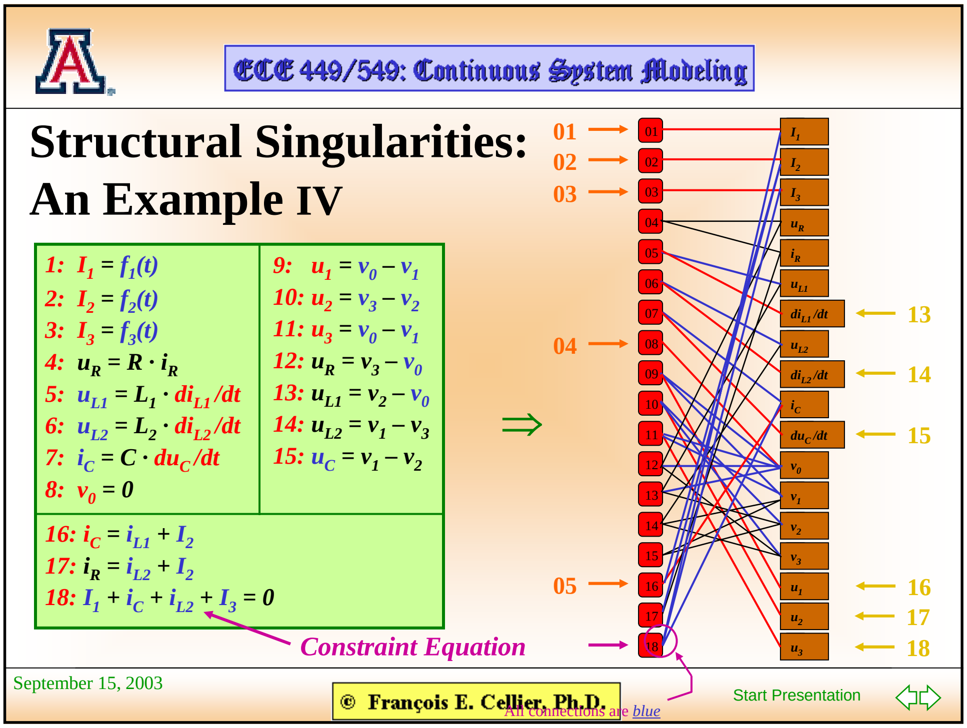

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Structural Singularities: An Example IV

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$

Constraint Equation

$\Rightarrow$

All connections are *blue*

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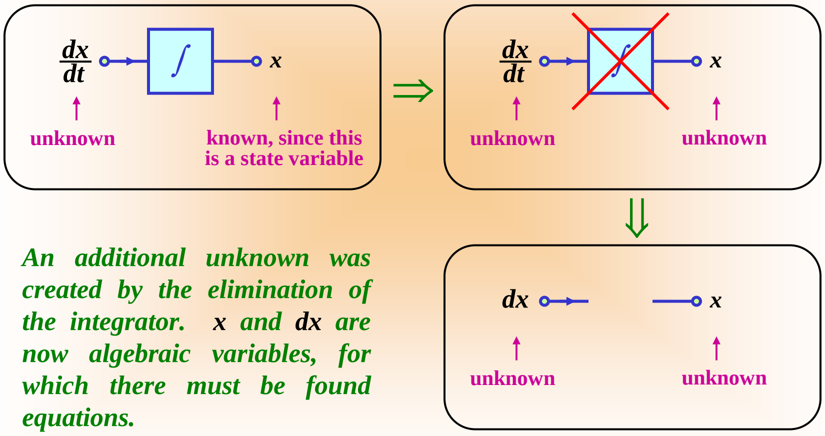
Coloring of the Structure Digraph

- The algorithm for ***coloring the structure digraph*** is completely analogous to the previously used method for making the equations causal.
- An implementation of the method by means of a computer program probably prefers the digraph, since this algorithm can directly be mapped onto data structures of conventional programming languages.
- For the human eye, the coloring of the equations may be more readable. For this reason, we shall continue, in the lecture, to color equations rather than digraphs.
- The vertical sorting can happen simultaneously by renumbering of the equations.

The Algorithm by Pantelides I

- As soon as a ***constraint equation*** has been found, this equation must be ***differentiated***.
- In the algorithm of Pantelides, the differentiated constraint equation is ***added*** to the set of equations.
- Consequently, the set of equations has now one equation too many.
- In order to re-equalize the number of equations and unknowns, one of the integrators that is associated with the constraint equation is being eliminated.

The Algorithm by Pantelides II



An additional unknown was created by the elimination of the integrator. x and dx are now algebraic variables, for which there must be found equations.

The Algorithm by Pantelides III

- When differentiating constraint equations, it can happen that additional new variables are being created, e.g. ***v*** → ***dv***, where ***v*** is an algebraic variable.
- Since ***v*** was already ***blue*** (otherwise, this would not have been a constraint equation), there already exists another equation to compute ***v***.
- This equation must also be differentiated.
- The differentiation of additional equations continues until no additional variables are being created.

The Algorithm by Pantelides : An Example I

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}\cancel{/dt}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $\boxed{I_1 + i_C + i_{L2} + I_3 = 0}$

eliminated integrator

$$\boxed{dI_1 + di_C + di_{L2} + dI_3 = 0}$$

newly introduced variables

The Algorithm by Pantelides : An Example II

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

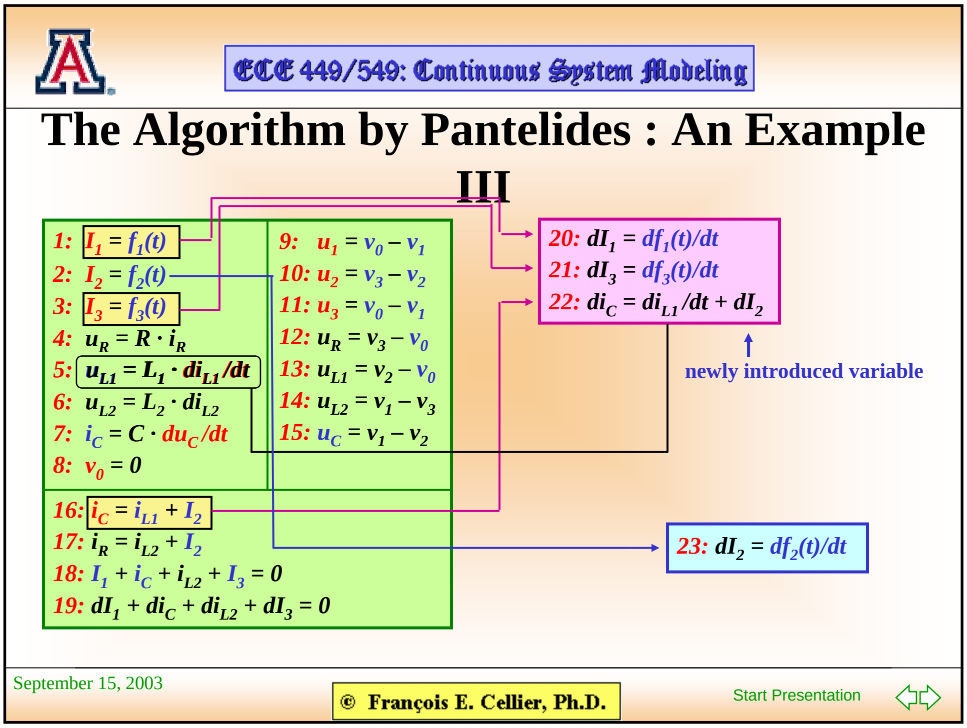
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

$\Rightarrow$

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$



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The Algorithm by Pantelides : An Example III

1: $\boxed{I_1 = f_1(t)}$
2: $I_2 = f_2(t)$
3: $\boxed{I_3 = f_3(t)}$
4: $u_R = R \cdot i_R$
5: $\boxed{u_{L1} = L_1 \cdot di_{L1}/dt}$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $\boxed{i_C = i_{L1} + I_2}$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$

newly introduced variable

23: $dI_2 = df_2(t)/dt$

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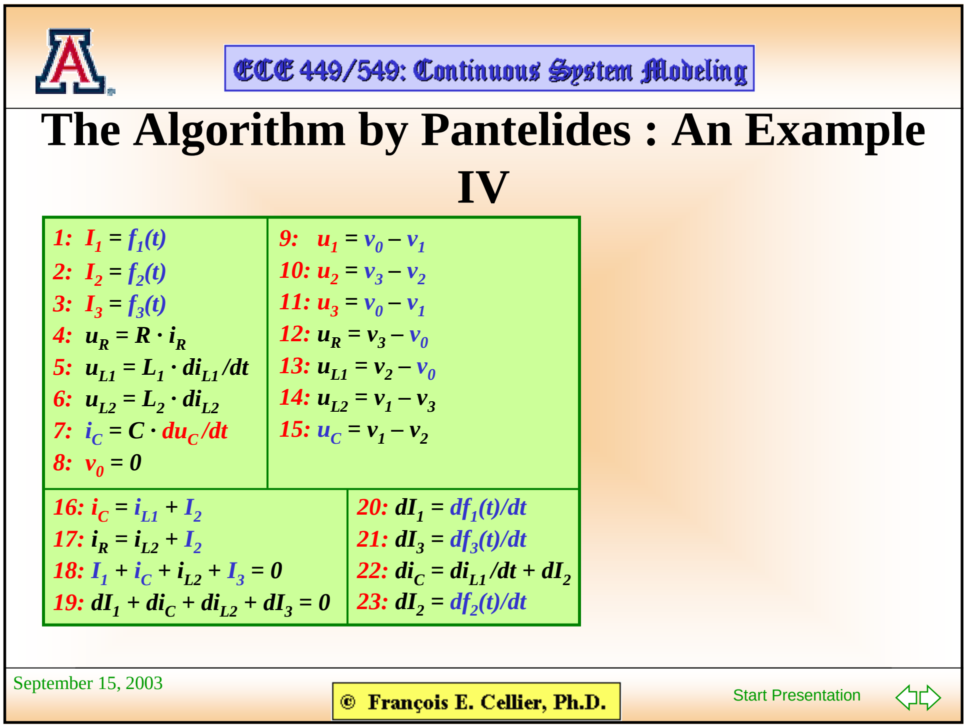

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The Algorithm by Pantelides : An Example IV

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

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The Algorithm by Pantelides : An Example V

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

The Algorithm by Pantelides : An Example VI

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

The Algorithm by Pantelides : An Example VII

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

The Algorithm by Pantelides : An Example VIII

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$

9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + \boxed{di_{L2}} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

Choice

There now exists an algebraically coupled system with 7 equations in 7 unknowns.

The Algorithm by Pantelides : An Example IX

1: $I_1 = f_1(t)$

2: $I_2 = f_2(t)$

3: $I_3 = f_3(t)$

4: $u_R = R \cdot i_R$

5: $u_{L1} = L_1 \cdot di_{L1}/dt$

6: $u_{L2} = L_2 \cdot di_{L2}$

7: $i_C = C \cdot du_C/dt$

8: $v_0 = 0$

9: $u_1 = v_0 - v_1$

10: $u_2 = v_3 - v_2$

11: $u_3 = v_0 - v_1$

12: $u_R = v_3 - v_0$

13: $u_{L1} = v_2 - v_0$

14: $u_{L2} = v_1 - v_3$

15: $u_C = v_1 - v_2$

16: $i_C = i_{L1} + I_2$

17: $i_R = i_{L2} + I_2$

18: $I_1 + i_C + i_{L2} + I_3 = 0$

19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

20: $dI_1 = df_1(t)/dt$

21: $dI_3 = df_3(t)/dt$

22: $di_C = di_{L1}/dt + dI_2$

23: $dI_2 = df_2(t)/dt$

Summary I

- First, we find a complete set of a-causal DAEs.
- The *graph coloring algorithm by Tarjan* is then applied to this set of DAEs.
- If an equation is found that is colored entirely in ***blue***, then the system is structurally singular.
- The structurally singular system is made non-singular by means of the ***algorithm by Pantelides***.
- It may be necessary to apply the Pantelides algorithm multiple times.

Summary II

- The ***graph coloring algoriths by Tarjan*** is now applied to the modified non-singular set of DAEs.
- If the algorithm stalls, the modified system now contains one or several ***algebraic loops***. The occurrence of algebraic loops after application of the Pantelides algorithm to a structurally singular system is quite common.
- The system can now be further processed. The ***tearing algorithm***, which has already been presented, is one possible approach to deal with algebraically coupled systems.

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- Pantelides, C.C. (1988), "The consistent initialization of differential-algebraic systems," *SIAM Journal Scientific Statistical Computation*, **9**(2), pp. 213-231.