Planar Mechanical Systems

- In this lecture, we shall deal with planar mechanical systems that can either translate or rotate in a two-dimensional space.
- We shall demonstrate the similarities between the mathematical descriptions of these systems and the electrical circuits discussed in the previous lecture.
- In particular, it will be shown that the symbolic formulae manipulation algorithms (sorting algorithms) that were introduced in the previous lecture can be applied to these systems just as easily and without any modification.

ECE 449/549: Continuous System Modeling

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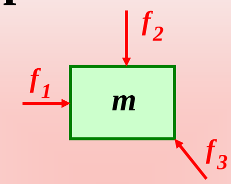
Start of Presentation

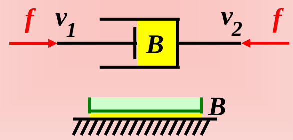
Linear Components of Translation

- Mass



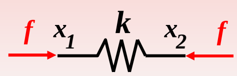
$$m \cdot a = \sum_{\forall i} (f_i)$$
$$\frac{dv}{dt} = a$$
$$\frac{dx}{dt} = v$$

- Friction



$$f = B \cdot (v_1 - v_2)$$

- Spring



$$f = k \cdot (x_1 - x_2)$$

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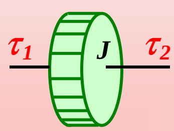
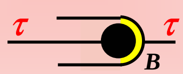
Linear Components of Rotation

- Inertia



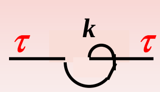
$$J \cdot \alpha = \sum_{\forall i} (\tau_i)$$
$$\frac{d\omega}{dt} = \alpha$$
$$\frac{d\theta}{dt} = \omega$$

- Friction



$$\tau = B \cdot (\omega_1 - \omega_2)$$

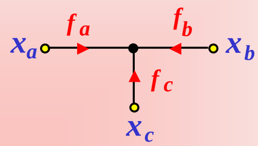
- Spring



$$\tau = k \cdot (\theta_1 - \theta_2)$$

Joints without Degrees of Freedom

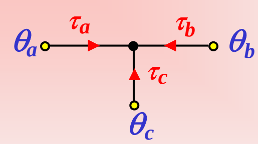
- Node (Translation)



$x_a = x_b = x_c$
$v_a = v_b = v_c$
$a_a = a_b = a_c$

$f_a + f_b + f_c = 0$

- Node (Rotation)



$\theta_a = \theta_b = \theta_c$
$\omega_a = \omega_b = \omega_c$
$\alpha_a = \alpha_b = \alpha_c$

$\tau_a + \tau_b + \tau_c = 0$

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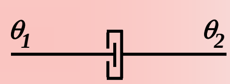
Joints with One Degree of Freedom

- Prismatic

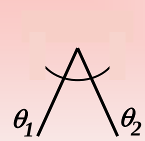
$x_1 \neq x_2$
$y_1 = y_2$
$\theta_1 = \theta_2$

- Cylindrical



$x_1 = x_2$
$y_1 = y_2$
$\theta_1 \neq \theta_2$

- Scissors



The D'Alembert Principle

- By introduction of an *inertial force*:

$$f_m = - m \cdot a$$

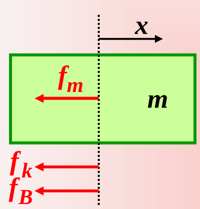
the second law of Newton:

$$m \cdot a = \sum_{\forall i} (f_i)$$

can be converted to a law of the form:

$$\sum_{\forall i} (f_i) = 0$$

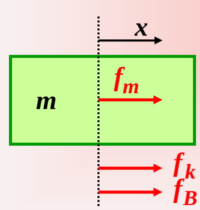
Sign Conventions



$$f_m = + \frac{d(m \cdot v)}{dt}$$

$$f_k = + k \cdot (x - x_{Neighbor})$$

$$f_B = + B \cdot (v - v_{Neighbor})$$

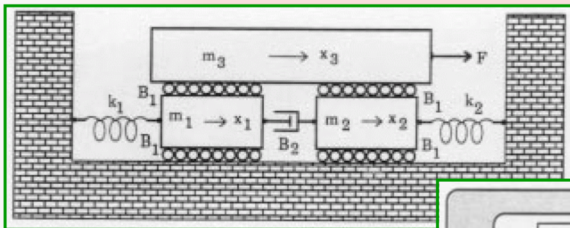


$$f_m = - \frac{d(m \cdot v)}{dt}$$

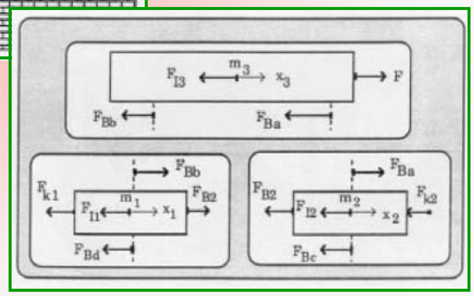
$$f_k = - k \cdot (x - x_{Neighbor})$$

$$f_B = - B \cdot (v - v_{Neighbor})$$

1. Example (Translation)



The system is being cut open between the individual masses, and cutting forces are introduced.



The D'Alembert principle can now be applied to each body separately.

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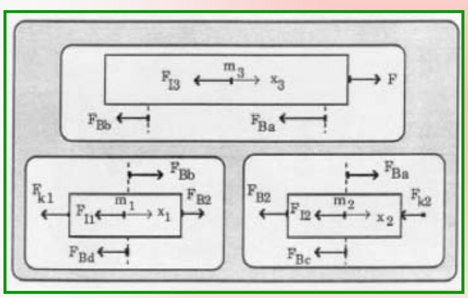
1. Example (continued)



$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$	$F_{Ba} = B_1 \cdot (v_3 - v_2)$
$\frac{dx_1}{dt} = v_1$	$F_{Bb} = B_1 \cdot (v_3 - v_1)$
$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$	$F_{Bc} = B_1 \cdot v_2$
$\frac{dx_2}{dt} = v_2$	$F_{Bd} = B_1 \cdot v_1$
$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$	$F_{B2} = B_2 \cdot (v_2 - v_1)$
$\frac{dx_3}{dt} = v_3$	$F_{k1} = k_1 \cdot x_1$
	$F_{k2} = k_2 \cdot x_2$

Horizontal Sorting I

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

$\Rightarrow$

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

Horizontal Sorting II

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

$\Rightarrow$

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

Horizontal Sorting III

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

$\Rightarrow$

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

Horizontal Sorting IV

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$
$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$
$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt} \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$F_{I2} = m_2 \cdot \frac{dv_2}{dt} \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$F_{I3} = m_3 \cdot \frac{dv_3}{dt} \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

$\Rightarrow$

$$F_{I3} = F(t) - F_{Ba} - F_{Bb}$$
$$F_{I2} = F_{Ba} - F_{Bc} - F_{B2} - F_{k2}$$
$$F_{I1} = F_{Bb} + F_{B2} - F_{Bd} - F_{k1}$$

$$\frac{dv_1}{dt} = F_{I1} / m_1 \qquad F_{Ba} = B_1 \cdot (v_3 - v_2)$$
$$\frac{dx_1}{dt} = v_1 \qquad F_{Bb} = B_1 \cdot (v_3 - v_1)$$
$$\frac{dv_2}{dt} = F_{I2} / m_2 \qquad F_{Bc} = B_1 \cdot v_2$$
$$\frac{dx_2}{dt} = v_2 \qquad F_{Bd} = B_1 \cdot v_1$$
$$\frac{dv_3}{dt} = F_{I3} / m_3 \qquad F_{B2} = B_2 \cdot (v_2 - v_1)$$
$$\frac{dx_3}{dt} = v_3 \qquad F_{k1} = k_1 \cdot x_1$$
$$F_{k2} = k_2 \cdot x_2$$

Sorting Algorithm

- The sorting algorithm operates exactly in the same way as for electrical circuits. It is totally independent of the application domain.

References

- Cellier, F.E. (1991), *Continuous System Modeling*, Springer-Verlag, New York, Chapter 4.