1. Electrical Circuit

Given the following electrical circuit:

- Capacitors: $C_1$, $C_2$, $C_3$
- Resistors: $R_1 = 20$, $R_2 = 100$
- Ground

The circuit diagram shows connections between these components.
Problems I

Find a set of horizontally and vertically sorted simulation equations describing this circuit.

Determine a state-space model for this circuit. The Voltage of the source is the input to the model. The power delivered by the Voltage source is the desired output variable.
\[ U_0 = f(t) \]
\[ u_{R1} = u_{C2} - u_{C1} \]
\[ i_1 = u_{R1} / R_1 \]
\[ u_{R2} = U_0 - u_{C2} \]
\[ i_0 = u_{R2} / R_2 \]
\[ i_2 = i_0 - i_1 \]
\[ du_{C1} / dt = i_1 / C_1 \]
\[ du_{C2} / dt = i_2 / C_2 \]

\[ du_{C1} / dt = i_1 / C_1 = u_{R1} / (R_1 \cdot C_1) = u_{C2} / (R_1 \cdot C_1) - u_{C1} / (R_1 \cdot C_1) \]

\[ du_{C2} / dt = i_2 / C_2 = i_0 / C_2 - i_1 / C_2 = u_{R2} / (R_2 \cdot C_2) - u_{R1} / (R_1 \cdot C_2) \]

\[ y = U_0 \cdot i_0 = U_0 \cdot u_{R2} / R_2 = U_0^2 / R_2 - U_0 \cdot u_{C2} / R_2 \]

The output equation is non-linear.

Problems II

Let \( C_2 \rightarrow 0 \). Describe what happens in the simulation equations, and interpret your findings by explaining what happens in the circuit itself.

\[ U_0 = f(t) \]
\[ u_{R1} = R_1 \cdot i_1 \]
\[ u_{R2} = R_2 \cdot i_0 \]
\[ i_1 = C_1 \cdot du_{C1} / dt \]
\[ i_2 = 0 \]
\[ i_0 = i_1 + i_2 \]
\[ u_{C2} = u_{C1} + u_{R1} \]
\[ U_0 = u_{C2} + u_{R2} \]

We now have an algebraic loop with five equations in five unknowns.
Two resistors in series produce an algebraic loop.

Problems III

- Make \( C_2 \neq 0 \), and this time, let \( R_2 \to 0 \). Describe what happens in the simulation equations, and interpret your findings by explaining what happens in the circuit itself.

\[
\begin{align*}
U_0 &= f(t) \\
\nu_{R1} &= R_1 \cdot i_1 \\
\nu_{R2} &= 0 \\
i_1 &= C_1 \cdot \frac{d\nu_{C1}}{dt} \\
i_2 &= C_2 \cdot \frac{d\nu_{C2}}{dt} \\
i_0 &= i_1 + i_2 \\
u_{C2} &= \nu_{C1} + \nu_{R1} \\
U_0 &= \nu_{C2} + \nu_{R2}
\end{align*}
\]

Constraint Equation

We now ended up with a structurally singular system.
C2 is now in parallel with the Voltage source. Hence the Voltage across that capacitor can no longer be used as an independent state variable. Thus, the system is structurally singular.

Problems IV

Make \( C_2 \neq 0 \) and \( R_2 \neq 0 \), and now, let \( R_1 \to 0 \). Describe what happens in the simulation equations, and interpret your findings by explaining what happens in the circuit itself.

\[
\begin{align*}
U_0 &= f(t) \\
u_{R1} &= 0 \\
u_{R2} &= R_2 \cdot i_0 \\
i_1 &= C_1 \cdot du_{C1}/dt \\
i_2 &= C_2 \cdot du_{C2}/dt \\
i_0 &= i_1 + i_2 \\
u_{C2} &= u_{C1} + u_{R1} \\
U_0 &= u_{C2} + u_{R2}
\end{align*}
\]

We again ended up with a structurally singular system.
The two capacitors are now in parallel. Two capacitors in parallel cause a structural singularity.
• A positive current through the coil ($R$, $L$) induces a force $F_{el}$ pulling the table down:

$$F_{el} = \psi \cdot i$$

• The displacement of the table produces an induced Voltage $u_i$ proportional to the velocity $v$ of the table, which opposes the applied Voltage $u(t)$:

$$u_i = \psi \cdot v$$

Problems

• Determine a state-space model for the system.
• Draw a block diagram for this system, which contains only integrators (no differentiators), and which shows $u(t)$ as input, and $x(t)$ as output.
Inputs are the variables $u$ and $F$. The output is the variable $x$.

The state variables are the outputs of the three integrators, hence the variables $i$, $v$, and $x$. 

The equations are:

\[ u = R \cdot i + L \cdot \frac{di}{dt} + u_i \]
\[ m \cdot \frac{d^2x}{dt^2} = F - k \cdot x - B \cdot \frac{dx}{dt} + F_{el} \]
\[ F_{el} = \psi \cdot i \]
\[ u_i = \psi \cdot \frac{dx}{dt} \]
\[ u = R \cdot i + L \cdot \frac{di}{dt} + u_i \]

\[ m \cdot \frac{d^2x}{dt^2} = F - k \cdot x - B \cdot \frac{dx}{dt} + F_d \]

\[ F_d = \psi \cdot i \]

\[ u_i = \psi \cdot \frac{dx}{dt} \]

\[ \frac{di}{dt} = \frac{(-R \cdot i - \psi \cdot v + u)}{L} \]

\[ \frac{dx}{dt} = v \]

\[ \frac{dv}{dt} = \frac{(\psi \cdot i - k \cdot x - B \cdot v + F)}{m} \]

\[
\begin{bmatrix}
\frac{di}{dt} \\
\frac{dx}{dt} \\
\frac{dv}{dt}
\end{bmatrix} =
\begin{bmatrix}
-R/L & 0 & -\psi/L \\
0 & 0 & 1 \\
\psi/m & -k/m & -B/m
\end{bmatrix}
\begin{bmatrix}
i \\
x \\
v
\end{bmatrix}
+ \begin{bmatrix}
1/L & 0 \\
0 & 0 \\
0 & 1/m
\end{bmatrix}
\begin{bmatrix}
u \\
F
\end{bmatrix}
\]
3. Mechanical System

- We wish to analyze the following system:

You may assume that it is an electrically driven car with a large DC battery. You may furthermore assume that the car has only a single wheel (simplification). There is linear friction in the motor itself, and between the wheel(s) and the road. You should also consider air friction (quadratic in the velocity of the car).
• Decompose the system into three separate parts:
  ♦ the electrical subsystem
  ♦ the rotational mechanics
  ♦ the translational mechanics

• describe separately the differential equations for each of the three subsystems, and add the coupling equations that connect them to each other.

The electrical side is simply an armature-controlled DC-motor:

\[ u_a = R_a \cdot i_a + L_a \cdot \frac{di_a}{dt} + u_i \]
The rotational side consists of two inertias with friction to ground:

\[
(J_m + J_w) \cdot \frac{d^2 \theta}{dt^2} = \tau_m - (B_m + B_w) \cdot \frac{d\theta}{dt} - \tau
\]

Conversion from electrical side

The translational side can be modeled using a simple free-body diagram:

\[
m \cdot \frac{d^2 x}{dt^2} = F - B \cdot \left( \frac{dx}{dt} \right)^2 - m \cdot g \cdot \sin(\alpha)
\]
The conversion equations can be written as:

1. **Electrical to rotational:**
   \[ \tau_m = \psi \cdot i_a, \quad u_i = \psi \cdot \frac{d\theta}{dt} \]

3. **Rotational to translational:**
   \[ \tau = r \cdot F, \quad x = r \cdot \theta \]

Reduce the equations to state-space form using:

- \( u = u_a \) (armature voltage of DC motor)
- \( x_1 = i_a \) (armature current of DC motor)
- \( x_2 = \theta \) (wheel angle)
- \( x_3 = \omega \) (wheel angular velocity)
- \( y = x \) (position of car along the road)
\[ u_a = R_a \cdot i_a + L_a \cdot \frac{di_a}{dt} + u_i \]

\[ \Rightarrow \frac{di_a}{dt} = \left( \frac{1}{L_a} \right) \cdot u_a - \left( \frac{R_a}{L_a} \right) \cdot i_a - \psi \cdot \omega \]

\[ (J_m + J_w) \cdot \frac{d^2 \theta}{dt^2} = \tau_m - (B_m + B_w) \cdot \frac{d\theta}{dt} - \tau \]

\[ \Rightarrow \frac{d\theta}{dt} = \omega \]

\[ \frac{d\omega}{dt} = \frac{\psi}{J_m + J_w} \cdot i_a - \frac{B_m + B_w}{J_m + J_w} \cdot \omega - \frac{r}{J_m + J_w} \cdot F \]

\[ m \cdot \frac{d^2 x}{dt^2} = F - B \cdot (\frac{dx}{dt})^2 - m \cdot g \cdot \sin(\alpha) \]

\[ \Rightarrow m \cdot r \cdot \frac{d\omega}{dt} = F - B \cdot r^2 \cdot \omega^2 - m \cdot g \cdot \sin(\alpha) \]

Now, we can eliminate the variable F from the last two equations:

\[ F = m \cdot r \cdot \frac{d\omega}{dt} + B \cdot r^2 \cdot \omega^2 + m \cdot g \cdot \sin(\alpha) \]

\[ \Rightarrow (J_m + J_w) \cdot \frac{d\omega}{dt} = \psi \cdot i_a - (B_m + B_w) \cdot \omega - m \cdot r^2 \cdot \frac{d\omega}{dt} - B \cdot r^3 \cdot \omega^2 - m \cdot g \cdot r \cdot \sin(\alpha) \]
\[(J_m + J_w) \cdot \frac{d\omega}{dt} = \psi \cdot i_a - (B_m + B_w) \cdot \omega - m \cdot r^2 \cdot \frac{d\omega}{dt} - B \cdot r^3 \cdot \omega^2 - m \cdot g \cdot r \cdot \sin(\alpha)\]

\[
\Rightarrow (J_m + J_w + m \cdot r^2) \cdot \frac{d\omega}{dt} = \psi \cdot i_a - (B_m + B_w) \cdot \omega - B \cdot r^3 \cdot \omega^2 - m \cdot g \cdot r \cdot \sin(\alpha)\]