1) **Differentiation Property:**

\[ \mathcal{Z}\{ \frac{\partial}{\partial a}(f(t,a)) \} = \frac{\partial}{\partial a}(\mathcal{Z}(f(z,a))) \]

9) **Convolution Property:**

\[ \mathcal{Z}\{ f(t) \ast g(t) \} = \mathcal{Z}(f(z)) \cdot \mathcal{Z}(g(z)) \]

\[ \mathcal{Z}\{ f(t) \cdot g(t) \} = \mathcal{Z}(f(z)) \ast \mathcal{Z}(g(z)) \]

\[ \Rightarrow \text{We can really compute with } \mathcal{Z}\text{-transforms exactly the same way as we are used to do with } \mathcal{L}\text{-transforms:} \]

\[ U(z) \quad G(z) \quad Y(z) \]

\[ \Rightarrow Y(z) = G(z) \cdot U(z) \]

However, this requires some words of interpretation.
\[ u(t^*) \] \[ \xrightarrow{T} \] \[ u^*(t) \] \[ \rightarrow \rightarrow \rightarrow \] \[ G(s) \] \[ \rightarrow \rightarrow \rightarrow \] \[ y(t) \]

\[ u^*(t) = \sum_{k=0}^{\infty} u(kT) \cdot \delta(t - kT) \]

Due to linearity, we can superimpose the individual direct responses.

\[ y(t) = \sum_{k=0}^{\infty} u(kT) \cdot g(t - kT) \]

If \( y(t) \) is only considered at the sampling times, we obtain:

\[ y(mT) = \sum_{k=0}^{\infty} u(kT) \cdot g(mT - kT) \]

\[ y(mT) = y^*(t) = g^*(t) \ast u^*(t) \]

\[ \Rightarrow y(z) = G(z) \cdot U(z) \]
To be able to use this convenient calculus, we need to sample (but not hold) all signals:

\[ u(t) \xrightarrow{T} u^*(t) \xrightarrow{G(s)} y(t) \xrightarrow{T} y^*(t) \]

\[ Y(z) = Z\{y^*(t)\} \]  
\[ = G(z) \cdot U(z) \]

where:  
\[ U(z) = Z\{u^*(t)\} \]

and:  
\[ G(z) = Z\{g^*(t)\} \]

where:  
\[ g(t) = L^{-1}\{G(s)\} \]

\[ \Rightarrow u \xrightarrow{T} G_1(s) \xrightarrow{T} G_2(s) \xrightarrow{T} y \]

\[ Y(z) = G_{12}(z) \cdot U(z) \]

where:  
\[ G_{12}(z) = Z\{g_{12}^*(t)\} \]

where:  
\[ g_{12}(t) = L^{-1}\{G_1(s) \cdot G_2(s)\} \]
\[ Y(z) = G_2(z) \cdot G_1(z) \cdot U(z) \]

and:
\[ G_2(z) \cdot G_1(z) = G_{12}(z) \]

If there is a \textit{2OH} in between, just add its transfer function in the Laplace domain to the plant. Thus:

\[ Y(z) = \hat{G}_2(z) \cdot \hat{G}_1(z) \cdot U(z) \]
A slight simplification is possible due to the shifting theorem:

\[ G_p(s) = \frac{1 - e^{-ts}}{s} \]

\[ \Rightarrow \hat{G}_1(s) = G_1(s) \cdot G_p(s) = \frac{G_1(s)}{s} - e^{-Ts} \frac{G_1(s)}{s} \]

\[ \Rightarrow \hat{g}_1(z) = \mathbb{Z}\{ e^{-z^1} \frac{G_1(s)}{s} \} - \mathbb{Z}\{ e^{-z^1} \frac{G_1(s)}{s} \} \]

\[ = \mathbb{Z}\{ e^{-z^1} \frac{G_1(s)}{s} \} - z^{-1} \cdot \mathbb{Z}\{ e^{-z^1} \frac{G_1(s)}{s} \} \]

\[ \Rightarrow \hat{g}_1(z) = (1 - z^{-1}) \cdot \mathbb{Z}\{ e^{-z^1} \frac{G_1(s)}{s} \} \]

For simplification, we are now going to define:

\[ \mathbb{Z}\{ e^{-z^1} G(s) \} = \mathbb{Z}\{ G(s) \} \]

\[ \Rightarrow \hat{g}_1(z) = (1 - z^{-1}) \cdot \mathbb{Z}\{ \frac{G(s)}{s} \} \]
Transformations Between the Data Representations Met So Far.

Questions: How do these different representations belong together? How can we find one of them if another one of them is given? How does the transformation work mathematically, and how does it work physically?

1) $g(t) \rightarrow g^*(t)$
a) mathematically: evaluate \( g(t) \) at the time instant, \( kT \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( g(0) )</td>
</tr>
<tr>
<td>( T )</td>
<td>( g(T) )</td>
</tr>
<tr>
<td>( 2T )</td>
<td>( g(2T) )</td>
</tr>
<tr>
<td>( 3T )</td>
<td>( g(3T) )</td>
</tr>
</tbody>
</table>

\[ \implies g^*(t) \]

b) physically: by a PAM device.

\[ g(t) \xrightarrow{\text{PAM}} g^*(t) \]

\[ g(t) \xrightarrow{\text{PAM}} g(t) \]

a) mathematically:

\[ g^*(t) \xrightarrow{\text{FFT}} G^*_jw \]  
\[ G^*_jw \xrightarrow{\text{low pass filter}} G(jw) \]
\[ G(jw) \xrightarrow{\text{FFT}^{-1}} g(t) \]
b) **physically:**

\[ g^*(t) \rightarrow \text{Low Pass Filter} \rightarrow g(t) \]

(3) \[ g(t) \rightarrow g_R(t) \]

9) **mathematically:**

Same as 0

b) **physically:**

\[ g(t) \rightarrow S/H \rightarrow g_R(t) \]

E.g.

Similar circuit diagram
\[ x_p = "1" : \text{ HOSFET switch is closed} \Rightarrow \]

The capacitor is charged to \[ g(kT) \] with a time constant \[ T_c = R \cdot C \]

\[ x_p = "0" : \text{ HOSFET switch is open} \Rightarrow \]

The amplifier works as an integrator with open input \[ \Rightarrow \text{ output stays constant} \]
\[ C \text{ is discharged through the (high) input impedance of the amplifier.} \quad (T_c^* = R_a \cdot C) \]
\[ T \gg \text{large} \]

Choose: \[ R \cdot C \ll p \ll T \ll R_a \cdot C \]

\[ \text{(4)} \]

\[ \begin{array}{c}
\frac{g_p(t)}{g(t)} \\
\hline
\end{array} \]

4) \text{mathematically:}

\[ \text{Same as (2)} \]

a) \text{mathematically:}

b) \text{physically:}

\[ \text{PAM} \rightarrow g^*(t) \rightarrow \text{Low Pass Filter} \rightarrow g(t) \]
5) \[ g(t) \rightarrow G(s) \]

a) mathematically:
\[ g(t) \rightarrow \mathcal{L} \rightarrow G(s) \]

\text{cf. ECE 441}

6) physically:

\( g(t) \) represents a \underline{signal}:

\[ g(t) \rightarrow \text{FFT} \rightarrow G(j\omega) \]

\( \text{Fast Fourier Transform is a discrete convolution. There exist very efficient special purpose chips for that purpose.} \)

\( g(t) \) represents the impulse response of a \underline{system}:

\text{Solution: Several ways are possible:}

(1)
\[ \int g(t) \rightarrow \text{System} \rightarrow \text{FFT} \rightarrow G(j\omega) \]

\( \uparrow \text{never very accurate} \)
\[ g(t) = \sin(\omega t) \]
\[ \dot{x}(t) = -\omega^2 \cdot \sin(\omega t) = -\omega^2 \cdot x(t) \]
Some companies are specialized in computerized equipment for this sort of signal processing (e.g. Hewlett Packard).

6) \[ G^*(s) \rightarrow G(z) \]
   simply replace \( e^{-Ts} \rightarrow z^{-1} \)

7) \[ G(z) \rightarrow G^*(z) \]
   simply replace \( z^{-1} \rightarrow e^{-Ts} \)

8) \[ G(s) \rightarrow G(z) \]

a) Use formulas directly as given before.

b) do partial fraction expansion of \( G(s) \), and map each term separately.

\[
\frac{k}{s+a} \quad \rightarrow \quad \frac{kz}{z-b}; \quad b = e^{-aT} \\
\frac{k}{(s+a)^2} \quad \rightarrow \quad \frac{kTbz}{(z-b)^2}; \quad b = e^{-aT}
\]

etc.
Best way is to use now a partial fraction expansion of \( G(t) \) but:

\[
\frac{k}{z+a} \rightarrow ?
\]

As seen from previous page, we know that:

\[
\frac{kz}{z+a} \rightarrow \frac{k}{s-b}; \quad b = \frac{1}{2} \ln(-a)
\]

**Trick:**

\[ \hat{G}(z) = \frac{G(t)}{z} \]

We decompose \( \hat{G}(z) \) by partial fraction expansion:

E.g.,

\[ \hat{G}(z) = \frac{A}{z+a} + \frac{B}{z+b} + \frac{C_1}{z+c} + \frac{C_2}{(z+c)^2} \]

\[ G(z) = z \cdot \hat{G}(z) = \frac{A z}{z+a} + \frac{B z}{z+b} + \frac{C_1 z}{z+c} + \frac{C_2 z}{(z+c)^2} \]

Now we know how to map back.