Stability of Discrete & Sampled Data Systems

When we sampled a continuous system:

\[ G(s) \xrightarrow{\text{sampling}} G^*(s) \]

we still have a "continues" system.

\[ \Rightarrow \text{The same properties hold as before:} \]

\[ G^*(s) \text{ is stable iff all "poles" (in } s \text{) are in the left half plane of } s \]

\[
\begin{array}{c}
\text{Im} \\
\text{Stable} \\
\text{Unstable}
\end{array}
\]

\[
\begin{array}{c}
\text{Re}
\end{array}
\]
Now, we introduced the abbreviation:

\[ z = e^{\frac{Ts}{2}}. \]

How does stability look in this new variables?

\[ s = \sigma + j\omega \]

\[ Ts = T(\sigma + j\omega) = T_0 \cdot e^{jT_\omega} \]

\[ \Rightarrow z = e^\sigma = e^{jT_\omega} = T_0 \cdot e^{jT_\omega} = 1 \cdot z \]

\[ |z| = e^{T_0} ; \angle z = T_\omega \]

Stability is granted if \[ \sigma < 0 \iff |z| < 1 \]

---

**Diagram:**

- **Stable** region:
  - Inside the circle
  - Below the imaginary axis

- **Unstable** region:
  - Outside the circle
  - Above the imaginary axis
Marginal stability in $\mathbb{S}$ is the imaginary axis:

$$\sigma = 0 \quad \longleftrightarrow \quad |z| = 1$$

$$\Rightarrow$$ For Lyapunov stability, we can tolerate single poles on the unity circle (in $z$), whereas multiple poles on the unity circle (in $z$) make the system unstable.

**Stability Reserve:**

If we want to guarantee that a certain settling time is not surpassed:

$$|z| \leq \frac{T}{T_s} \quad \longleftrightarrow \quad |z| \leq e^{-\frac{t}{T_s}}$$
Real poles:

5% Overshooting:
We usually ask not more than 5% overshooting.

=> curves of constant damping are less convenient in the z-domain.
"Good" pole locations:

Construction of the logarithmic spiral:

let \[ \Omega = \frac{2\pi}{T} \]

\[ \angle z = \frac{1}{T} \omega = 2\pi \cdot \frac{3}{2} \]

45° damping \[ \Rightarrow |\theta| \equiv |\omega| \]

\[ \angle z = 180° = \pi \Rightarrow \omega = \frac{\Omega}{2} = |\theta| \]

\[ |z| = e^{\frac{-T \cdot \theta}{2}} = e^{-\frac{\pi}{2}} = e^{-\pi} \]

\[ |z| = e^{-\frac{\pi}{2}} = e^{-\pi} = e^{-\pi} = 0.0432 \]
\[ \angle \beta = 90^\circ = \frac{\pi}{2} \quad \Rightarrow \quad \omega = \frac{\omega}{4} = 1 \circ \]

\[ |z| = e^{-\frac{\pi}{4}} = e^{-\frac{\pi}{4}} = e^{-\frac{\pi}{4}} = 0.2079 \]

\[ \angle \beta = 45^\circ = \frac{\pi}{4} \quad \Rightarrow \quad |z| = e^{-\frac{\pi}{4}} = 0.4559 \]

\[ \angle \beta = 135^\circ = \frac{3\pi}{4} \quad \Rightarrow \quad |z| = e^{-\frac{3\pi}{4}} = 0.0948 \]

\[ \text{etc.} \]

Assume: \(|\sigma| \geq 4\) \(\Rightarrow T_s \leq 1\text{sec}\)

\[ \Rightarrow |z| \leq e^{-4T} \]

; Assume \(T = 0.1 \Rightarrow |z| \leq 0.6703 \)

Stability reserve = \(f(T)\)

5% overshoot \(\neq f(T)\)

Good pole locations
Problem: In the past, we mostly looked at real poles.

\[ s = -a \leftrightarrow z = e^{-at} \]

This mapping is unique both ways.

However:

\[ s = -a \pm jb \leftrightarrow z = e^{-at} \sqrt{\frac{1}{1 \pm jb}} \]

The angle is $2\pi$-periodic, thus:

![Diagram showing mapping](Image)

\[ 2\pi \uparrow \]

not unique
\[ \hat{A} = P \cdot A \cdot P^{-1} = \Lambda \]

Pole locations don't change.
(eigenvalues are invariant to similarity transformations)

\[ \Rightarrow \hat{f} = e^{\hat{A}T} \Rightarrow \hat{f} = e^{\Lambda T} = Q \cdot \hat{f} \cdot Q^{-1} \]

\[ \Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \rightarrow \quad \Lambda^T = \begin{bmatrix} e^{\lambda_1 T} & e^{\lambda_2 T} & \cdots & 0 \\ 0 & e^{\lambda_2 T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n T} \end{bmatrix} \]

If the continuous system has its poles at \( \Xi \in \mathbb{C} \Rightarrow \) the discretized system (with or without ZOH) has its poles at \( \Xi \cdot e^{\lambda T} \).

\[ \Rightarrow \text{If } G(s) \text{ is stable } \Rightarrow G(z) \text{ will be stable also (with or without ZOH).} \]

**However:** Given \( A \), we can find \( \hat{f} \) in a unique manner. The opposite is unfortunately not true.
What does this mean in practice?

Assume: \( \omega T = 2\pi \frac{\omega}{\Omega} = \phi \)

\[ \Rightarrow \omega = \{\phi, \Omega, 2\Omega, \ldots \} \]

By sampling with a sampling period of \( T \), the two signals cannot be distinguished at all.

Remember: Sampling Theorem: \( \Omega \geq 2\omega_{\text{max}} \)

This is not satisfied for \( \omega = \Omega \)

\[ \Rightarrow \] We always assume that the engineer was reasonable enough to satisfy the sampling theorem.
In that case, the mapping from \( z \rightarrow s \) is also unique.

\[
\begin{align*}
\text{Symmetry:} \\
\text{We remember that complex poles in the } \Sigma \text{-plane always appear as conjugate complex pairs as the real axis is a symmetry axis.}
\end{align*}
\]

\[
\begin{align*}
S_1 &= -a + jb \\
\implies \angle S_1 &= T \theta \\
S_2 &= -a - jb \\
\implies \angle S_2 &= -T \theta
\end{align*}
\]
Complex poles appear also in the $\mathbb{C}$ plane always as conjugate complex pairs $\Rightarrow$ the real axis is a symmetry axis.

Approximation by dominant poles:

The poles with the largest distance from the origin (inside the unity circle) are the dominant poles of a stable system.

Example:
\[ r = e^{\sigma T} \Rightarrow \sigma = \frac{1}{T} \ln(r) \]

\[ \varphi = \omega T \Rightarrow \omega = \frac{\varphi}{T} \]

\[ \omega_0 = \sqrt{\sigma^2 + \omega^2} \Rightarrow J = \left| \frac{10}{3\varphi} \right| \]

\[ J = \cos(\alpha) = \left| \frac{10}{3\varphi} \right| \]

J: damping factor
Remember: The approximation by dominant poles was justified if all other poles were "sufficiently" further to the left.

→ No other poles had a smaller damping.

→ Dominant poles have no zeros close by.

Otherwise, consider all poles and simulate.

- Approximation is bad, as 3rd pole is too close.

- Approximation is bad, as other pole have not enough damping.
In the \( \mathbb{Z} \)-plane, the first condition is easy to check.

-> Then may not exist other poles with similar distance to the origin.

\[ \text{Im} \quad \mathbb{Z} \]
\[ \text{Re} \quad \times \]

\( \bigcirc \) dominant poles

\( \Rightarrow \) approximation is bad
\( \Rightarrow \) 3rd pole is too close.

The second condition requires to sketch the logarithmic spiral that leads through the dominant poles:

\[ \text{Im} \quad \mathbb{Z} \]
\[ \text{Re} \quad \times \]

\( \bigcirc \) dominant poles

\( \Rightarrow \) approximation is bad as other poles are outside the logarithmic spiral.
As the algebraic structure of continuous & discrete systems is the same, the definitions for stability remain the same as well:

1) BIBO-stability: (stability with respect to the input):

⇒ A system is BIBO-stable iff all poles of \( G(s) \) lie inside the unity circle:

\[
|z_i| < 1
\]

2) Lyapunov-stability: (stability with respect to the initial conditions):

⇒ A system is Lyapunov-stable iff all eigenvalues of \( F \) lie inside or on the unity circle:

\[
|\lambda_i| \leq 1
\]
and all eigenvalues on the unity circle are single:

$$| \lambda_i | = 1 \quad \Rightarrow \quad n_i = 1$$

**Example:**

$$\begin{align*}
\begin{bmatrix}
X(k+1) \\
Y(k)
\end{bmatrix} &= \begin{bmatrix}
\phi & 1 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
X(k) \\
Y(k)
\end{bmatrix} \\
Y(k) &= \begin{bmatrix}
1 & \phi
\end{bmatrix}
\begin{bmatrix}
X(k) \\
Y(k)
\end{bmatrix}
\end{align*}$$

$$\Rightarrow i \begin{bmatrix}
1
\end{bmatrix} \Rightarrow \lambda_i = 1 \text{ on unity circle.}$$

$$\Rightarrow X(i) = 1 \quad \text{for all } i$$

$$Y(i) = 1 \quad \text{for all } i$$

$$\Rightarrow \text{stable}.$$
System has double pole at \(+1\)

\[
F = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 1 \quad m_1 = 2
\]

\[
x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow x(2) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \Rightarrow x(3) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}
\]

\[
\Rightarrow x(4) = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \text{ etc.}
\]

\[
y(0) = 1 \quad y(1) = 0 \quad y(2) = -1 \quad y(3) = -2 \quad \text{etc.}
\]

\[
\Rightarrow y \text{ goes to } -\infty
\]

\[
\Rightarrow \text{System is indeed unstable.}
\]

\[
\text{Warning: It is not sufficient to check one initial condition to ensure stability. E.g., the last example with } x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ does not go to infinity.}
\]
For a system to be **stable**, it must both be **BIBO-stable** and **Lyapunov-stable**.

**Example:**

\[
\begin{align*}
\dot{x}(k+1) &= x(k) + u(k) \\
y(k) &= x(k) + u(k)
\end{align*}
\]

\(x(0) = x_0\)

is Lyapunov-stable (input \(u(k) = 0\))

but is not BIBO-stable (\(x_0 = 0\))

which can be easily verified by computing the step response:

\(x(0) = 0 \Rightarrow x(1) = 1 \Rightarrow x(2) = 2 \Rightarrow \ldots\)

\(y(0) = 1 \Rightarrow y(1) = 2 \Rightarrow y(2) = 3 \Rightarrow \ldots\)

\(y \rightarrow +\infty\)
Example:

\[
\begin{bmatrix}
\phi & 1 \\
-1 & 2.5
\end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} -1 & 0.5 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}
\]

\[
F = \begin{bmatrix} \phi & 1 \\
-1 & 2.5
\end{bmatrix} \Rightarrow \lambda_1 = 0.5, \quad \lambda_2 = 2
\]

\Rightarrow \text{System is not Lyapunov stable.}

\[
G(z) = 1 + \frac{0.5z - 1}{z^2 - 2.5z + 1}
\]

\[
= \frac{z^2 - 2.5z + 1 + 0.5z - 1}{z^2 - 2.5z + 1} = \frac{z^2 - 2z}{z^2 - 2.5z + 1}
\]

\[
G(z) = \frac{z(z - 2)}{(z - 0.5)(z - 2)} = \frac{z}{z - 0.5}
\]

\Rightarrow \text{System is BIBO stable.}