PHYSICAL REVIEW LETTERS

VOLUME 83 | 25 OCTOBER 1999 | NUMBER 17

On the Radiating and Nonradiating Components of Scalar, Electromagnetic, and Weak Gravitational Sources

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(Received 16 April 1999)

We derive a new description of two complementary classes of scalar, vector, or tensor sources, namely, nonradiating (NR) sources and sources that lack a NR part, i.e., "purely radiating" sources. The class of purely radiating sources is shown to be exactly the class of solutions—within the source's support—of the homogeneous form of the associated partial differential equation relating the sources to their fields; i.e., purely radiating sources are themselves fields. We show that this result is related to the well-known reciprocity principle and establish a new definition of a localized NR source based on that principle. A broad class of physically relevant sources, all of which possess a NR part, is identified. The role of NR sources in absorption of radiation and energy storage is addressed.

PACS numbers: 03.50.−z

It has been known for some time that localized sources to the scalar wave equation and to Maxwell's equations exist which do not radiate [1,2]. Such sources, referred to as nonradiating (NR) sources, generate vanishing fields outside their support [2] which prevents them from interacting with nearby objects by means of their fields. Work on NR sources dates back to Sommerfeld, Herglotz, Hertz, Ehrenfest, and Schott (see references in [1]) who studied these objects in connection with electron and atom models. NR sources have also appeared extensively in inverse source/scattering theories as the null space of the source-to-field mapping [3,4]. NR sources have been studied very recently in connection with strings [5]. In this Letter, we characterize in novel ways both NR sources and sources that lack a NR part, i.e., "purely radiating" sources.

Consider a general complex-valued scalar, vector, or tensor "source-field" system (ρ, ψ) described by a linear, second-order partial differential equation (PDE) $L\psi(\mathbf{x}) = \rho(\mathbf{x})$, where $\mathbf{x} \in R^n$ denotes the relevant space or space-time coordinates, L is a linear second-order PD operator, and ρ and ψ are "sources" and "fields" to that equation, respectively. The field ψ produced by a source ρ can be expressed as the Green function integral $\psi(\mathbf{x}) = \int d^n x' \times \rho(\mathbf{x}')G_0(\mathbf{x} \,|\, \mathbf{x}')$, where G_0 is the Green function associated with L and the given boundary conditions. Table I lists physically realizable examples of these source-field systems, indicating the relevant coordinates, field, source, and PD operator associated with each one.

We define the Hilbert space U of L_2 sources ρ supported in a simply connected source region D and assign to it the inner product $\langle \rho_1, \rho_2 \rangle = \int_D d^n x\, \rho_1^*(\mathbf{x})\rho_2(\mathbf{x})$, where * denotes the complex conjugate. The following result is a trivial generalization of known results on NR scalar and electromagnetic sources [2,6]: Any NR source $\rho_{\rm NR} \in U$, i.e., any L_2 source of support D whose field vanishes for $\mathbf{x} \notin D$, must admit the representation

$$\rho_{\rm NR}(\mathbf{x}) = L\psi_{\rm NR}(\mathbf{x})\,, \qquad (1)$$

where $\psi_{\rm NR}$, a function supported in D, is exactly the field produced by that NR source. Now, because of the assumed L_2 nature of the NR source in (1) and, in particular, its lack of single-layer and higher-order singularities on the boundary ∂D of D, one finds that $\psi_{\rm NR}$ must possess compact support in D [so that $\psi_{\rm NR}(\mathbf{x}) = 0$ on ∂D] in addition to continuous first partial derivatives on the boundary ∂D of D. The above-stated *NR boundary conditions* were derived first by Gamliel *et al.* [7] in the context of the inhomogeneous Helmholtz equation in 3D space and rederived later by Berry *et al.* [5] for the 1D case. They have also

0031-9007/99/83(17)/3345(5)$15.00

TABLE I. Scalar, vector, and tensor equations describing source-field systems of interest. Indicated are the relevant space or space-time coordinates, source, field, and PD operator associated with each equation. In the table $\mathbf{x} = (x, y, z)$ and t denote position in space and time, respectively. The 3D form of the equation for a forced string in an elastic medium (a Klein-Gordon equation with an inhomogeneous term) is relevant to scalar meson fields. In the line corresponding to the linearized Einstein equations, given here in the Lorentz (or Hilbert) gauge, $\psi_{\mu\nu}$, $T_{\mu\nu}$, and λ are the gravitational potential, the energy-momentum tensor, and Einstein's gravitational constant, respectively.

Coordinates	Field (ψ)	Source (ρ)	PD Operator (L)	Description
$(\mathbf{x}, t)$	$\psi(\mathbf{x}, t)$	$q(\mathbf{x}, t)$	$\Box \equiv \nabla^2 - c^{-2}\partial_t^2$	Wave equation
$\mathbf{x}$	$\psi(\mathbf{x})$	$q(\mathbf{x})$	$\nabla^2 + k^2$	Helmholtz equation
$(\mathbf{x}, t)$	$\mathbf{E}(\mathbf{x}, t)$	$-c^{-2}\partial_t \mathbf{J}(\mathbf{x}, t)$	$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times + c^{-2}\partial_t^2$	Vector wave equation
(x, t)	$\psi(x, t)$	$F(x, t)$	$\partial_x^2 - c^{-2}\partial_t^2 - m^2$	Forced string in an elastic medium
$x_\mu = (\mathbf{x}, ict)$	$\psi_{\mu\nu}$	$2\lambda T_{\mu\nu}$	$\Box$	Linearized Einstein equations

appeared in two recent papers dealing with NR sources and their fields [8,9].

Even though gravitational NR sources have received little attention in the literature, there are well-known examples of such NR sources (see, e.g., [10], pp. 974–979). The simplest example is provided by a pulsating (or even collapsing) spherically symmetric source. In particular, the exterior metric of such an object (e.g., a star), namely, Schwarzchild's metric [as required by Birkhoff's theorem (see, e.g., [10], p. 843)], is well known to be insensitive to the object's radial fluctuations. This result is actually an exact one; i.e., it is not a weak field approximation. It corresponds to the lack of gravitational monopole radiation. Other well-known exact gravitational NR sources are the mass-dipole and the magnetic-dipole. An analogous situation arises in electromagnetic theory where spherically symmetric and, in general, longitudinal current distributions do not radiate.

We show below that in order for a scalar, electromagnetic, or weak gravitational source $\rho \in U$ to lack a NR part, in the sense that $\langle \rho_{\mathrm{NR}}, \rho \rangle = 0$ for all NR sources $\rho_{\mathrm{NR}} \in U$, then

$$L\rho(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in D \tag{2}$$

(the boundary ∂D of D excluded), i.e., ρ must be a free-field, truncated within its support D. One deduces from this result and the unique decomposition of an L_2, localized source into a radiating and a NR part (see, e.g., the projection theorem discussion in [11]) that, ultimately, the radiating parts, i.e., *the sources of wave radiation are, themselves, fields.* We also provide below a connection between this result and the reciprocity principle, along with a new definition of a NR source based on that principle.

It follows from (1) and the generalized Green theorem that

$$\langle \rho_{\mathrm{NR}}, \rho \rangle = \int_D d^n x\, \psi_{\mathrm{NR}}^*(\mathbf{x}) \tilde{L} \rho(\mathbf{x}), \tag{3}$$

where $\tilde{L}$ is the adjoint of the PD operator L, as defined, e.g., in [12], Chap. 15. If the PD operator L is self-adjoint, as is, in fact, the case for all the source-field systems listed in Table I, then it follows from (3) that the orthogonality condition $\langle \rho_{\mathrm{NR}}, \rho \rangle = 0$ will hold for arbitrary ψ_{NR} if and only if (2) holds [note that, because of the vanishing of ψ_{NR} on ∂D for $\rho_{\mathrm{NR}} \in U$, the integral in (3) actually involves *only* the interior of the source region D]. Expression (2) thus defines the sought-after necessary and sufficient condition for an L_2 source ρ of support D to be purely radiating, i.e., to lack a NR component. This condition tells us that the class of all such purely radiating sources is exactly the class of solutions of the homogeneous form of the PDE governing the corresponding source-field system, in the interior of the source's support. For example, for a time-harmonic, space-dependent electromagnetic source-field system $(\mathbf{J}, \mathbf{E})$ with a suppressed $e^{-i\omega t}$ time dependence, expression (2) takes the form

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{J}(\mathbf{x}) - k^2 \mathbf{J}(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in D \tag{4}$$

(the boundary ∂D of D excluded) where $k = \omega/c$. The validity of (4), and of the scalar counterparts of (2) for $L = \nabla^2 + k^2$ and $L = \nabla^2 - c^{-2}\partial_t^2$, can be verified, for special cases, directly from inverse source problem results and, in particular, from results on the so-called minimum L_2 norm/minimum energy sources presented in [4,11,13,14].

The general result (2) leads to yet another previously unknown result: It can be shown via standard Green function techniques that no (nontrivial) source of compact support D and vanishing first partial derivatives on the boundary ∂D of D exists that obeys the requirement (2). This interesting consequence of (2) can be stated as follows: *Let ρ be a source of compact support D and vanishing first partial derivatives on the boundary ∂D of D, then ρ has a NR part.* We have thus found an important, broad class of physically relevant sources (to be referred to as well-behaved sources) all of which possess a NR part. By being applicable to a very tangible class of sources, this result emphasizes, in a previously unnoticed way, the undeniable existence of NR sources in nature. Of course, this *does not* mean that all L_2, localized sources must possess a NR part: purely radiating L_2, localized sources (i.e., sources in the orthogonal complement of the null space of the source-to-field propagator) must exist; otherwise there would be no wave radiation at all. To fully appreciate these issues, one must bear in mind that, actually, there are more general classes of sources other than the well-behaved ones that do not obey the necessary and sufficient condition (2) and, therefore, also have a NR part. These points are illustrated with an example below.

We establish next a connection between the results above and a new definition of a NR source based on the concept of reciprocity. For this purpose, we note first that the result (3) actually holds for all localized (not necessarily L_2) NR sources $\rho_{\rm NR}$. One then concludes from (3) that, in general, any localized scalar, electromagnetic, or weak gravitational NR source $\rho_{\rm NR}$, of support D, must be necessarily orthogonal to all solutions of (2), in *both* the interior and on the boundary ∂D of the NR source's support D. Actually, this condition is also sufficient: *A general source confined within D is NR if and only if it obeys the orthogonality relation* $\langle v, \rho_{\rm NR}\rangle = 0$ *with respect to all solutions* v*, in its support* D*, of the homogeneous form of the PDE of the associated source-field system,* i.e., $Lv(\mathbf{x}) = 0$ for $\mathbf{x} \in D$. To show sufficiency one simply notes that $LG_0^*(\mathbf{x}', \mathbf{x}) = 0$ if $\mathbf{x}' \notin D$ and $\mathbf{x} \in D$ so that $\psi_{\rm NR}(\mathbf{x}') = \int_D d^n x\, \rho_{\rm NR}(\mathbf{x}) G_0(\mathbf{x}', \mathbf{x}) = 0$ for $\mathbf{x}' \notin D$. This automatically characterizes localized NR sources as *noninteractors*. In other words, localized NR sources do not absorb power from incident fields. The physical explanation of this property is provided by the well-known reciprocity principle: *If a source does not radiate, then it does not receive either, and vice versa.* The NR component of a localized source is thus both invisible to external observers and noninteracting to external fields. The question then arises naturally as to whether localized NR sources possess any physical significance at all. The answer is positive: A localized NR source stores (nontrivial) field energy. It thus plays a role in a system's energy budget and dynamics.

We elaborate on our results with the aid of an electromagnetic example in 1D space. The chosen example deals with the basic problem of wave radiation and reception in a transmission line system (equivalently, a 1D plane wave system) driven by uniformly distributed sources/loads. It illustrates in a very basic context not only some of the general results derived above but also other closely related ideas. We use Gaussian units and $c = 1$.

Let $\mathbf{J}(\mathbf{x}) = J(x)\hat{\mathbf{z}}$ be a homogeneous current distribution (with a suppressed $e^{-i\omega t}$ time dependence) localized within $D = [-a, a]$ so that

$$J(x) = \begin{cases} 1 & \text{if } x \in D\,, \\ 0 & \text{else.} \end{cases} \qquad (5)$$

The z component of the electric field produced by this source, $E(x)$, related to $J(x)$ by

$$(d^2/dx^2 + k^2)E(x) = -i\omega J(x)\,,$$

is found after a straightforward analysis to be

$$E(x) = \begin{cases} \frac{i}{k}[e^{ika}\cos(kx) - 1], & \text{if } |x| \le a\,, \\ -a\,\mathrm{sinc}(ka)e^{ik|x|}, & \text{else,} \end{cases}$$

where $\mathrm{sinc}(\cdot) \equiv \sin(\cdot)/(\cdot)$. Now we note that the source $J(x)$ does not obey the relevant particular form of (2), i.e.,

$$(d^2/dx^2 + k^2)J(x) \neq 0 \quad \text{if } x \in (-a, a)\,.$$

Thus, according to the discussion above, this source must possess a NR part.

It follows from standard linear inversion theory [11] that $J(x)$ can be uniquely decomposed into the sum of a radiating and a NR part, $J_R(x)$ and $J_{\rm NR}(x)$, respectively, where $J_R(x)$ is the minimum L^2 norm source whose field $E_R(x) = E(x)$ for $x \notin D$. After some manipulations along the lines of [11,13] one obtains

$$J_R(x) = \frac{2\,\mathrm{sinc}(ka)}{[\mathrm{sinc}(2ka) + 1]}\, J(x)\cos(kx)$$

and, consequently, the nontrivial NR component $J_{\rm NR}(x) = J(x) - J_R(x)$. Figure 1 shows plots of the radiating part $J_R(x)$ of $J(x)$ versus x/a, parametrized by ka. The radiating part $J_R(x)$ of $J(x)$ is a standing wave truncated within the source's support D, as expected from the general result (4). It possesses compact support in D only if $ka = (n + 1/2)\pi$, $n = 0, 1, \ldots$, i.e., it is a resonant wave solution at those frequencies. It vanishes if $ka = n\pi$, $n = 1, 2, \ldots$; i.e., a homogeneous source oscillating at these quantized frequencies is entirely NR. The size of the smallest NR homogeneous source is then $2a = \lambda$, where λ is the wavelength of the field. Figure 2 shows plots of the real and reactive power of the J-E self-interaction as a function of ka/π. Also shown in Fig. 2 is a plot of the NR contribution to the reactive power. The real power of the J-E self-interaction equals the time-averaged radiated power and is contributed only by the radiating part

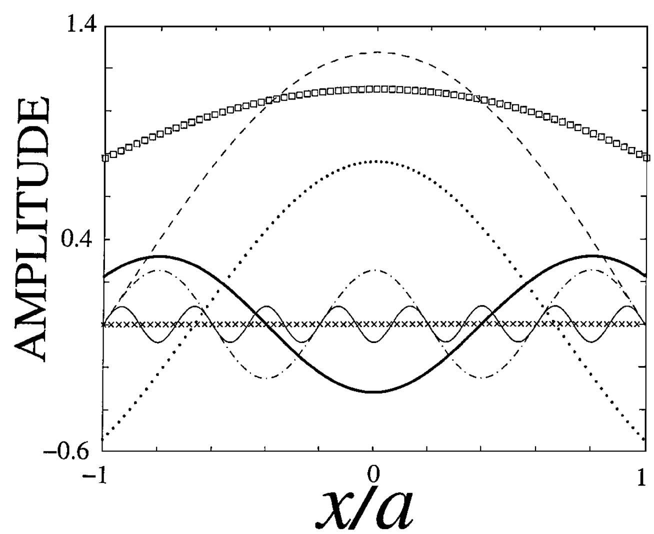

FIG. 1. Plots of the radiating part $J_R(x)$ of $J(x)$ versus x for various values of ka: $ka = 0.25\pi$ (squares); $ka = 0.5\pi$ (dashed line); $ka = 0.75\pi$ (dotted line); $ka = \pi$ (cross); $ka = 1.25\pi$ (bold solid line); $ka = 2.5\pi$ (dashed-dotted line); $ka = 7.5\pi$ (regular solid line).

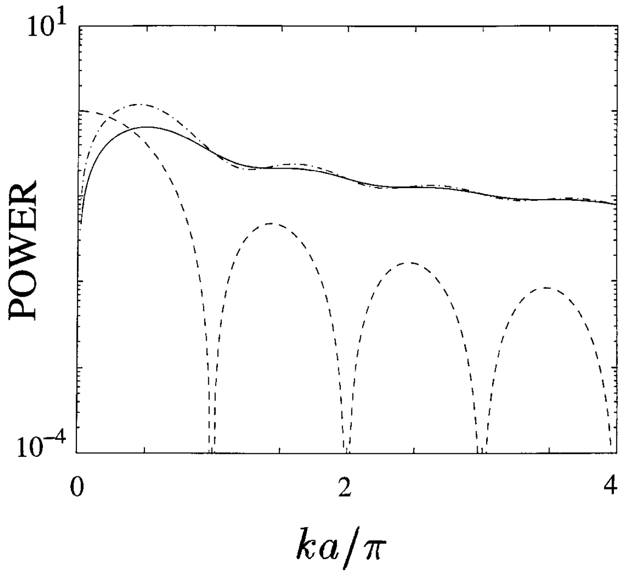

FIG. 2. Dashed line: Time-averaged power radiated by the source $J(x)$ versus ka/π (the value at $ka \simeq 0$ is used as a reference for normalization); this is also the (time-averaged) radiation-reaction power of this source and the interaction power of two nonoverlapping homogeneous sources of the form Eq. (5). Solid line: Total reactive power due to $J(x)$. Dashed-dotted line: Reactive power due to $J_{\mathrm{NR}}(x)$.

$J_R(x)$ of $J(x)$, as expected. On the other hand, the reactive energy is contributed by both the radiating and NR parts. The reactive contribution of the radiating part decays rapidly for $ka \gtrsim \pi$. The NR component $J_{\mathrm{NR}}(x)$ of $J(x)$ has an associated nontrivial (time-averaged) stored energy. The NR modes defined by $ka = n\pi$, $n = 1, 2, \ldots$ represent states of pure energy storage. The (time-averaged) energy stored in the electric and magnetic fields, W_e and W_m, respectively, associated with these NR modes are found after a straightforward calculation to be $W_e = 3a^2/(2n\pi)^2$ and $W_m = a^2/(2n\pi)^2$. Thus, the NR component of a system represents a nontrivial, noninteracting accumulation of energy and is therefore relevant to the system's energy balance and dynamics. Finally, we also show in Fig. 2 the real interaction power of two nonoverlapping sources J, J' of the form Eq. (5) (with the support D' of J' being entirely outside D). Only the radiating parts (i.e., the wave components) of these sources interact. The nonabsorbing nature of the NR parts was verified additionally by calculating their interactions with incident plane waves.

In this Letter, we arrived at a new description of both NR and purely radiating sources, i.e., sources that lack a NR part. Our general results apply to all source-field systems described by a second-order PDE. Particular attention was given to the scalar, electromagnetic, and weak gravitational source-field systems listed in Table I. We showed that in order for a source to be purely radiating, i.e., to lack a NR component, it must obey the homogeneous form of the PDE governing the associated source-field system. This condition was shown to be also sufficient for a source to lack a NR part; it therefore represents a new definition of the class of all such purely radiating sources. This also shows that the ultimate sources of wave radiation are themselves fields. An important, broad class of sources (well-behaved sources) all of which possess a NR component was identified which demonstrates the ever-presence of an unobservable and nonabsorbing component in a large number of physically relevant objects interacting with neighboring objects through fields. We also derived a new definition of a NR source based on the concept of reciprocity. We saw then that the NR source/free-field orthogonality established here in the context of new definitions of both NR and purely radiating sources can be regarded from a physical standpoint as a manifestation of a NR source's null interactivity, both in transmission (radiation) and in reception (absorption). Purely radiating sources represent situations of optimum interaction energy. NR sources, on the contrary, elude self- and external interactions and represent situations of optimum energy storage. The NR component of a system is therefore relevant to the system's dynamics. For example, an extended charged harmonic oscillator vibrating in the vicinity of one of its allowed NR modes could be "attracted" by that energetically stable mode and stay there until perturbed by nonelectromagnetic interactions (e.g., mechanical forces). Thus, noninteractivity does not nullify a NR source's importance. In the same vein, we conjecture that by virtue of their noninteractivity, hence, their potential stability to radiative interactions, NR sources could play a factor in the evolution of complex particle-field systems, e.g., in cosmological evolution. For the sake of simplicity, when referring to weak gravitational fields we did so in the context of a particular gauge, i.e., the Lorentz (or Hilbert) gauge. We are currently working on the true force field counterpart of the weak gravitational potential analysis considered above and plan to report elsewhere on the results of that analysis.

This work was supported by the NSF under Grant No. ECS-9900246 and by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under Grant No. F49620-96-1-0039. E. A. M. acknowledges enlightening discussions with Professor Anthony J. Devaney related to this work.

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