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Inverse source problem with regularity constraints: normal solution and nonradiating source components

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Abstract. The use of regularity constraints in formulating the scalar inverse source problem (ISP) is investigated. Two kinds of regularity constraints are considered: compact supportness in a given source region, and normal differentiability on the boundary of that region. Normal solutions (minimum L_2 norm solutions) to the ISP for square-integrable (L_2) scalar sources with and without the above-mentioned regularity constraints are derived and compared. The (generally nontrivial) nonradiating parts of the corresponding normal solutions are evaluated.

Keywords: Inverse source problem, nonradiating sources

1. Introduction

A problem of considerable interest for the object reconstruction branches of the wave disciplines (such as optics and acoustics) is the so-called inverse source problem (ISP) [1–3]. In its usual form, the ISP consists of deducing a source of known support, say D, from knowledge of its generated field outside D. Different aspects of this problem have been investigated by Müller [4], Moses [5, 6], Friedlander [7], Bleistein and Cohen [1], Hoenders [8], Devaney [9], Devaney and Porter [2, 3], LaHaie [10, 11], Carter and Wolf [12], Wolf [13], Bertero [14], Marengo and Devaney [15] and Marengo *et al* [16, 17]. Attention has been given to both the scalar and electromagnetic cases, for both deterministic and random sources. The focus has been on the fundamental nonuniqueness question [18] and *a priori* constraints that may render the ISP unique. The latter include the constraint of minimizing the solution's L_2 norm, which has lead to the so-called minimum energy solutions [2, 3]. Applications include inverse scattering-based surveys [18–21], holographic imaging [2, 3, 22] and antenna design [4]. The time-dependent ISP with far-field data has also received attention recently as an analogue of the limited-view Radon inversion problem that arises in the formalism of computerized tomography [16].

This work is concerned with the ISP for deterministic square-integrable (L_2) scalar sources ρ contained within a spherical volume $V = \{\boldsymbol{r} \in R^3 | r \leqslant a\}$ of radius a, with centre at the coordinate origin. The focus is on the use of *a priori* regularity (smoothness) constraints in formulating the ISP. The formulation is based on the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) \tag{1}$$

in three-dimensional space. The field $\psi(\boldsymbol{r})$ generated by a source $\rho(\boldsymbol{r})$ is then given by the familiar outgoing Green function integral

$$\psi(\boldsymbol{r}) = \int \mathrm{d}^3r' \rho(\boldsymbol{r}') \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{2}$$

We review first the ISP for general $L_2(V)$ sources, and examine later the ISP for $L_2(V)$ sources with additional regularity constraints. We are particularly interested in $L_2(V)$ sources ρ that possess *compact support* in the source volume V, for which $\rho(\boldsymbol{r}) = 0$ on the boundary $\partial V = \{\boldsymbol{r} \in R^3 | r = a\}$ of V. We shall also consider the more regular class of $L_2(V)$ sources of compact support V whose normal derivatives also vanish on the boundary ∂V of V. The results presented in this paper provide, to our knowledge, the first investigation of ISPs with such regularity constraints. Extension to yet more regular classes of sources, although not to be considered here, follows lines similar to those provided here for the cases above.

A unique solution to the usual ISP for general $L_2(V)$ sources, without additional constraints, cannot be obtained due to the presence of nontrivial nonradiating (NR) sources [1, 7, 23] localized within the source volume V. In particular, the field produced by a localized NR source vanishes outside the source's support. It then follows that, without additional pieces of information about the field and/or the source, the NR source components of a source cannot be deduced from knowledge of its exterior field. The usual ISP admits a unique solution if one imposes the additional constraint of minimizing the source's L_2 norm. The solution in question is the usual minimum energy solution [2, 3], also known as 'the normal solution' in linear inversion language [14]. Physical interpretations of these normal solutions have been given in [2, 3, 22] in the context of generalized holography.

Normal solutions to the usual ISP are orthogonal to all L_2 NR sources confined within the source region, i.e., they lack a NR part [2, 3]. It is shown in this paper that, in contrast, by imposing additional regularity constraints to the usual ISP, one can actually extract NR source components of the unknown source. In particular, we derive expressions for the normal solutions corresponding to ISPs with regularity constraints, along with their (generally nontrivial) NR source contributions. We thus examine a means of extracting NR source components of an unknown radiating (or scattering) object that is known *a priori* to be reasonably well behaved in a sense specified by given regularity constraints. We then also illustrate the use of *a priori* information in formulating the ISP. The latter question had been investigated, for other forms of *a priori* information, by Moses [5, 6] and Bleistein and Cohen [1]. In addition, the present analysis also corroborates, for the special case of L_2 sources contained in the spherical volume V, a recent result derived in [17] which states that any L_2 source of compact support having vanishing normal derivatives on the boundary of its support must possess a NR part. The general theory, based on spherical harmonics and Bessel functions, is illustrated with a spherically symmetric source example.

2. The inverse source problem for $L_2(V)$ sources

In this section, we formulate, by means of a general linear inversion formulation, the ISP for L_2 sources ρ of support $V = \{\boldsymbol{r} \in R^3 | r \leqslant a\}$ (such that $\rho(\boldsymbol{r}) = 0$ if $r > a$). The general approach developed in this section will be specialized in section 3 to $L_2(V)$ sources with various degrees of regularity on the boundary $\partial V = \{\boldsymbol{r} \in R^3 | r = a\}$ of the source volume V. In the present section, we also consider the unique decomposition of a source into its radiating and NR parts in the Hilbert space X of $L_2(V)$ sources with the defined inner product

$$(\rho, \rho')_X = \int_V \mathrm{d}^3 r \rho^*(\boldsymbol{r})\rho'(\boldsymbol{r}) \tag{3}$$

where * denotes the complex conjugate. The general source decomposition results developed here will find use in section 4 in connection with a spherically symmetric source example. We will then illustrate how the NR source components become, in general, increasingly noticeable as one imposes stricter source regularity properties. In particular, it will be shown that normal solutions to ISPs with regularity constraints contain, in general, NR source components in the Hilbert space X.

It is well known [24] that for $r > a$ the field $\psi(\boldsymbol{r})$ radiated by a source $\rho \in X$ can be expressed in the multipole expansion form

$$\psi(\boldsymbol{r}) = \mathrm{i}k \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{l,m} h_l^{(1)}(kr) Y_{l,m}(\hat{\boldsymbol{r}}) \tag{4}$$

where $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$, $h_l^{(1)}(\cdot)$ is the spherical Hankel function of the first kind and order l (as defined in [24], p 740), and $Y_{l,m}(\cdot)$ is the spherical harmonic of degree l and order m (as defined in [24], p 99). The expansion coefficients $g_{l,m}$ in equation (4) are the multipole moments and are defined by the inner products

$$g_{l,m} = (\psi_{l,m}, \rho)_X \tag{5}$$

where

$$\begin{aligned} &\psi_{l,m}(\boldsymbol{r}) = 4\pi H(a-r) j_l(kr) Y_{l,m}(\hat{\boldsymbol{r}}) \\ &l = 0, 1, \ldots; \quad m = -l, -l+1, \ldots, l \end{aligned} \tag{6}$$

where $j_l(\cdot)$ is the spherical Bessel function of the first kind and order l (as defined in [24], p 740) and $H(\cdot)$ is Heaviside's unit step function.

The field for $r > a$ defined by equation (4) is uniquely determined by the multipole moments. Because of this, in the following we formulate the ISP of deducing the $L_2(V)$ source with minimum L_2 norm that is consistent with a given data vector $\boldsymbol{g} = \{g_{l,m}\}$ having entries $g_{l,m}$. We assume the latter to be square-summable so that $\sum_{l=0}^{\infty} \sum_{m=-l}^{l} |g_{l,m}|^2 < \infty$. We also define the discrete Hilbert space Y of all such square-summable data vectors and assign to it the inner product

$$(\boldsymbol{g}, \boldsymbol{g}')_Y = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{l,m}^* g'_{l,m}. \tag{7}$$

To address the ISP in this framework, we define, by using equation (5), the linear source-to-data vector mapping

$$P\rho = \boldsymbol{g} \tag{8}$$

which assigns to each source $\rho \in X$ a data vector $\boldsymbol{g} \in Y$ according to the rule

$$(P\rho)_{l,m} = (\psi_{l,m}, \rho)_X. \tag{9}$$

The class of L_2 NR sources of support V is exactly the null space $N(P)$ of the linear mapping P [1, 23].

In the following, we shall assume the field for $r > a$ and, in particular, its corresponding data vector $\boldsymbol{g}$, to be realizable from $L_2(V)$ sources. In mathematical language, we require the data vector $\boldsymbol{g}$ to be in the range of the linear source-to-data vector mapping associated with the source space $L_2(V)$. The range in question has been defined explicitly in [17] by using the so-called Picard conditions [14] that apply to this ISP. In particular, the range $R(P)$ of P consists of the data vectors $\boldsymbol{g}$ that obey the necessary and sufficient condition

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} |g_{l,m}|^2 / \sigma_l^2 < \infty \tag{10}$$

where

$$\begin{aligned} \sigma_l^2 &\equiv (4\pi)^2 \int_0^a \mathrm{d}r r^2 j_l^2(kr) \\ &= 8\pi^2 a^3 [j_l^2(ka) - j_{l-1}(ka) j_{l+1}(ka)]. \end{aligned} \tag{11}$$

Under this condition, the normal solution to the ISP, corresponding to the unique source $\hat{\rho}$ of minimum L_2 norm (minimum energy) associated with a given data vector $\boldsymbol{g}$, is defined by the pseudoinverse of P and is given by [14]

$$\hat{\rho} = P^{\dagger}(PP^{\dagger})^{-1}\boldsymbol{g} \tag{12}$$

where $P^{\dagger}$ is the adjoint of the linear mapping P, defined by

$$(P\rho, \boldsymbol{g})_Y = (\rho, P^{\dagger}\boldsymbol{g})_X. \tag{13}$$

The latter is found from equations (3), (7), (9), (13) to be given by

$$(P^{\dagger}\boldsymbol{g})(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}\psi_{l,m}(\boldsymbol{r}). \tag{14}$$

It is not hard to show by using equations (6), (9), (11), (12), (14) and the orthogonality property of the spherical harmonics that

$$\begin{aligned}\hat{\rho}(\boldsymbol{r}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}\psi_{l,m}(\boldsymbol{r})/\sigma_l^2 \\ &= 4\pi H(a-r)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m} j_l(kr) Y_{l,m}(\hat{\boldsymbol{r}})/\sigma_l^2. \end{aligned} \tag{15}$$

The normal solution $\hat{\rho}$ given by equation (15) consists of a source-free multipole expansion, over the truncated spherical wavefunctions $\psi_{l,m}$, with multipole moments $g_{l,m}/\sigma_l^2$. Expression (15) can be shown to be in the form of the usual singular system representation of the normal solution associated with the linear mapping P [1, 14]. The terms σ_l^2 are known to decay exponentially fast for $l > ka$, confirming the ill-posed nature of the ISP [14].

The ISP results presented above can be used to uniquely decompose a source ρ in the Hilbert space X into its radiating and NR source components. In particular, it is a well-established fact [2, 3, 14] that any source $\rho \in X$ can be uniquely decomposed into the sum $\rho = \hat{\rho} + \rho_{NR}$ of a radiating and a NR part, $\hat{\rho}$ and ρ_{NR} respectively, where $\hat{\rho}(\boldsymbol{r})$ is exactly the normal solution in equation (15) corresponding to the data vector produced by the given ρ. The normal solution to the ISP formulated above, in which we imposed no regularity constraints, thus lacks a NR part in the Hilbert space X. In section 3, we will depict a different scenario for ISPs with regularity constraints. We will show then that normal solutions to such ISPs do contain, in general, NR source components in the Hilbert space X. We will also illustrate an interesting result derived in [17] which states that any L_2 source of compact support having vanishing normal derivatives on the boundary of its support must possess a NR part.

3. The inverse source problem for $L_2(V)$ sources with regularity constraints

In this section, we consider L_2 sources ρ that are compactly supported in the spherical volume $V = \{r \in R^3 | r \leqslant a\}$ (such that $\rho(\boldsymbol{r}) = 0$ if $r \geqslant a$). Any such source must admit a representation of the form

$$\rho(\boldsymbol{r}) = \sum_{L=0}^{\infty}\sum_{M=-L}^{L} q_{L,M}(r) Y_{L,M}(\hat{\boldsymbol{r}}) \tag{16}$$

where $q_{L,M}(r)$ is an r-dependent function that can be expanded in the Fourier–Bessel series form

$$q_{L,M}(r) = \sum_{n=0}^{\infty} a(n, L, M; v)\rho_{n;v}(r) \tag{17}$$

where

$$\rho_{n;v}(r) = \frac{\sqrt{2/a^3}}{|j_{v+1}(\beta_{v,n})|} H(a-r) j_v(\beta_{v,n} r/a) \tag{18}$$

where v is an arbitrary non-negative integer. The parameters $\beta_{v,n}$ in equation (18) are consecutive zeros of the spherical Bessel function $j_v(\cdot)$, i.e. $j_v(\beta_{v,n}) = 0, n = 0, 1, 2, \ldots$. The functions $\rho_{n;v}(r)$ are orthonormal over V. The expansion coefficients $a(n, L, M; v)$ associated with a given $q_{L,M}(r)$ are then

$$a(n, L, M; v) = (\rho_{n;v}, q_{L,M})_X. \tag{19}$$

In deriving these results we have made use of the completeness and orthogonality of the spherical harmonics $Y_{L,M}(\cdot)$ over the unit sphere, the completeness of the spherical Bessel functions $j_v(\beta_{v,n} r/a)$ for fixed non-negative integer v and variable index n over the interval $[0, a]$ for functions that vanish at $r = a$ (see equation (11.51) of [25]), and the orthogonality property of the set of ordinary Bessel functions $J_v(\beta_{v,n} r/a)$ for fixed non-negative integer v and variable index n in the r-interval $[0, a]$ (see equation (11.168) of [25]).

Now, the non-negative integer v in equations (16)–(19) is arbitrary. For our purposes, the particular choice $v = L \geqslant 0$ will prove to be especially useful. With this choice, expressions (17)–(19) become

$$q_{L,M}(r) = \sum_{n=0}^{\infty} a(n, L, M; L)\rho_{n;L}(r) \tag{20}$$

where

$$\rho_{n;L}(r) = \frac{\sqrt{2/a^3}}{|j_{L+1}(\beta_{L,n})|} H(a-r) j_L(\beta_{L,n} r/a) \tag{21}$$

and

$$a(n, L, M; L) = (\rho_{n;L}, q_{L,M})_X. \tag{22}$$

The above results will enable us to formulate the ISP for L_2 sources that possess compact support in the source volume V by means of a linear inversion formalism analogous to that employed in section 2 for general $L_2(V)$ sources. In the following, we shall denote as $L_2^{(0)}(V) \subset L_2(V)$ the class of L_2 sources that are compactly supported in V. From the general results of section 2 and equations (16), (20)–(22), the ISP for $L_2^{(0)}(V)$ sources can be shown to reduce to finding the source expansion coefficients $a(n, L, M; L)$ from knowledge of the multipole moments $g_{l,m}$ of the source's exterior field. The relevant linear source expansion vector-to-data vector mapping is determined by substituting from equations (16), (20)–(22) into (5), (6). With these observations, we proceed next to evaluate the normal solution to the ISP investigated here (with the additional compact supportness constraint).

By analogy with the procedure employed in section 2 for general $L_2(V)$ sources, we introduce the discrete Hilbert space U of source expansion vectors $\boldsymbol{a} = \{a(n, L, M; L)\}$ that are square-summable so that

$$\sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L} |a(n, L, M; L)|^2 = \int_V \mathrm{d}^3 r |\rho(\boldsymbol{r})|^2 < \infty, \tag{23}$$

and assign to it the inner product

$$(\boldsymbol{a}, \boldsymbol{a}')_U = \sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L} a^*(n, L, M; L) a'(n, L, M; L). \tag{24}$$

We also recall here the definition, given in connection with equation (7), of the discrete Hilbert space Y of square-summable data vectors $\boldsymbol{g}$ having multipole moment entries $g_{l,m}$. With these Hilbert space definitions, we introduce next, also by analogy with the procedure employed in section 2, the linear source expansion vector-to-data vector mapping

$$\mathcal{P}\boldsymbol{a} = \boldsymbol{g} \tag{25}$$

which assigns to each source expansion vector $\boldsymbol{a} \in U$ a data vector $\boldsymbol{g} \in Y$ according to the rule

$$(\mathcal{P}\boldsymbol{a})_{l,m} = \sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L} a(n, L, M; L)(\psi_{l,m}, \rho_{n;L} Y_{L,M})_X. \tag{26}$$

Now, because of the orthogonality of the spherical harmonics,

$$(\psi_{l,m}, \rho_{n;L} Y_{L,M})_X = \delta_{l,L}\delta_{m,M}\alpha_{l,n} \tag{27}$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta and

$$\begin{aligned}\alpha_{l,n} &\equiv \frac{4\pi\sqrt{2/a^3}}{|j_{l+1}(\beta_{l,n})|}\int_0^a \mathrm{d}r r^2 j_l(kr) j_l(\beta_{l,n} r/a) \\ &= \frac{2\sqrt{2}\pi^2 a^{-1}(\beta_{l,n}/k)^{1/2}}{|j_{l+1}(\beta_{l,n})|[k^2 - (\beta_{l,n}/a)^2]} J_{l+1/2}(ka) J'_{l+1/2}(\beta_{l,n})\end{aligned} \tag{28}$$

where $J_l(x) = \sqrt{2x/\pi}\, j_{l-1/2}(x)$ and $J'_l(x) = \frac{\mathrm{d}}{\mathrm{d}x} J_l(x)$, if ka is not a zero of $j_l(\cdot)$, and

$$\alpha_{l,n} = \frac{\sqrt{2/a^3}}{4\pi|j_{l+1}(\beta_{l,n'})|}\sigma_l^2 \delta_{n,n'}, \tag{29}$$

with σ_l^2 given by equation (11), if ka is the n'th zero of $j_l(\cdot)$. In evaluating the integral defining $\alpha_{l,n}$ in equation (28), we have made use of the first Lommel integral (see [25], p 594). By substituting from equation (27) into (26) one obtains

$$(\mathcal{P}\boldsymbol{a})_{l,m} = \sum_{n=0}^{\infty} a(n, l, m; l)\alpha_{l,n}. \tag{30}$$

We also introduce the adjoint $\mathcal{P}^\dagger$ of the linear mapping $\mathcal{P}$. By using $(\mathcal{P}\boldsymbol{a}, \boldsymbol{g})_Y = (\boldsymbol{a}, \mathcal{P}^\dagger\boldsymbol{g})_U$, one obtains from equations (7), (24), (28)–(30) the result

$$(\mathcal{P}^\dagger\boldsymbol{g})(n, L, M; L) = g_{L,M}\alpha_{L,n}. \tag{31}$$

We can now derive an expression for the source $\hat{\rho}^{(0)}$ of minimum L_2 norm, among all L_2 sources that possess compact support in the source volume V, whose generated field coincides with a given data field for $r > a$. In particular, the source expansion vector $\hat{\boldsymbol{a}}$ corresponding to the normal solution $\hat{\rho}^{(0)}$ is defined by the pseudoinverse of $\mathcal{P}$:

$$\hat{a}(n, L, M; L) = \mathcal{P}^\dagger(\mathcal{P}\mathcal{P}^\dagger)^{-1}\boldsymbol{g}. \tag{32}$$

The linear operator $\mathcal{P}\mathcal{P}^\dagger : Y \to Y$ is found from equations (30), (31) to be defined by

$$(\mathcal{P}\mathcal{P}^\dagger\boldsymbol{g})_{l,m} = g_{l,m}\sum_{n=0}^{\infty}\alpha_{l,n}^2. \tag{33}$$

It then follows that

$$[(\mathcal{P}\mathcal{P}^\dagger)^{-1}\boldsymbol{g}]_{l,m} = g_{l,m}\left[\sum_{n=0}^{\infty}\alpha_{l,n}^2\right]^{-1}. \tag{34}$$

By using equations (31), (32), (34) one obtains the result

$$\hat{a}(n, L, M; L) = g_{L,M}\alpha_{L,n}\left[\sum_{n'=0}^{\infty}\alpha_{L,n'}^2\right]^{-1}. \tag{35}$$

The normal solution $\hat{\rho}^{(0)}$ corresponding to a given data vector $\boldsymbol{g}$ can be expressed directly in the configuration space by using equations (16), (20)–(22). One obtains

$$\begin{aligned}\hat{\rho}^{(0)}(\boldsymbol{r}) &= \sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\hat{a}(n, L, M; L)\rho_{n;L}(r) Y_{L,M}(\hat{\boldsymbol{r}}) \\ &= \sqrt{2/a^3} H(a-r)\sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L} g_{L,M}\alpha_{L,n} \\ &\quad\times\left[|j_{L+1}(\beta_{L,n})|\sum_{n'=0}^{\infty}\alpha_{L,n'}^2\right]^{-1} j_L(\beta_{L,n} r/a) Y_{L,M}(\hat{\boldsymbol{r}}).\end{aligned} \tag{36}$$

It is not hard to see that whenever ka is a zero of the spherical Bessel function $j_l(\cdot)$, expression (36) with $\alpha_{l,n}$ given by equation (29) reduces to the normal solution in equation (15) corresponding to the ISP without regularity constraints. This is to be expected since then, the (generally nonregular) normal solution in equation (15) is, by itself, compactly supported (regular) in the source region V. The effect of the additional compact supportness constraint addressed in this section becomes visible, however, for the general case when ka is not a zero of the spherical Bessel function. The corresponding normal solution is then defined by equation (36) with $\alpha_{l,n}$ given by equation (28). The associated nontrivial NR part $\hat{\rho}_{NR}^{(0)}$ of $\hat{\rho}^{(0)}$ in the Hilbert space X is

$$\hat{\rho}_{NR}^{(0)} = \hat{\rho}^{(0)} - \hat{\rho}. \tag{37}$$

The L_2 norm of the normal solution defined by equation (36) is

$$\begin{aligned}\int_V \mathrm{d}^3 r|\hat{\rho}^{(0)}(\boldsymbol{r})|^2 &= (\hat{\boldsymbol{a}}, \hat{\boldsymbol{a}})_U \\ &= \sum_{n=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L}|g_{L,M}|^2\alpha_{L,n}^2\left[\sum_{n'=0}^{\infty}\alpha_{L,n'}^2\right]^{-2}.\end{aligned} \tag{38}$$

Furthermore, the condition

$$\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|g_{l,m}|^2\alpha_{l,n}^2\left[\sum_{n'=0}^{\infty}\alpha_{l,n'}^2\right]^{-2} < \infty \tag{39}$$

can be shown to be exactly the Picard condition defining the range $R(\mathcal{P})$ of the linear mapping $\mathcal{P}$. In particular, equation (39) defines the class of valid data vectors $\boldsymbol{g} = \{g_{l,m}\}$ associated with the source space $L_2^{(0)}(V)$.

3.1. Regularity constraints for the normal derivatives

The previous general formulation can be used for ISPs with yet stricter regularity constraints. We consider next the ISP for L_2 sources that are compactly supported in V and possess vanishing normal derivatives on the boundary ∂V of the source volume V. For the sake of brevity, we shall refer to sources obeying all the above-imposed localization and regularity properties as 'well-behaved sources'.

The ISP for well-behaved sources can be addressed by means of an approach similar to that employed above for the $L_2^{(0)}(V)$ sources. However, we must define first an orthonormal basis in terms of which all well-behaved sources can be expanded. To derive such a basis, we note that any well-behaved source must admit a representation of the form

$$\rho(\boldsymbol{r}) = \sum_{L=0}^{\infty}\sum_{M=-L}^{L} s_{L,M}(r)Y_{L,M}(\hat{\boldsymbol{r}}) \tag{40}$$

where

$$s_{L,M}(r) = \sum_{n=0}^{\infty} a(n, L, M; v)\rho_{n;v}(r) \tag{41}$$

where $\rho_{n;v}(r)$ is defined by equation (18), and the expansion coefficients $a(n, L, M; v)$ are constrained so as to ensure $\frac{\mathrm{d}}{\mathrm{d}r}s_{L,M}(r)|_{r=a} = 0$. In particular,

$$\sum_{n=0}^{\infty} a(n, L, M; v)\frac{\mathrm{d}}{\mathrm{d}r}\rho_{n;v}(r)|_{r=a} = 0, \tag{42}$$

i.e.,

$$\sqrt{2/a^3}H(a-r)\sum_{n=0}^{\infty} a(n, L, M; v)|j_{v+1}(\beta_{v,n})|^{-1} \times \frac{\mathrm{d}}{\mathrm{d}r}[j_v(\beta_{v,n}r/a)]|_{r=a} = 0 \tag{43}$$

where v is, as before, an arbitrary non-negative integer, which we choose to be $v = L \geqslant 0$. Equations (40)–(43) are used below to establish an orthonormal basis for well-behaved sources.

Consider the sequence $\{u_{p,L,M;L}(\boldsymbol{r})\}$, $p = 1, 2, \ldots$, $L = 0, 1, \ldots$, $M = -L, -L+1, \ldots, L$, of well-behaved functions

$$u_{p,L,M;L}(\boldsymbol{r}) = \sum_{n=0}^{p} \nu^{(p)}(n; L)\rho_{n;L}(r)Y_{L,M}(\hat{\boldsymbol{r}}) \tag{44}$$

with the expansion coefficients $\nu^{(p)}(n; L)$ subjected to the 'well-behavedness' constraint equation

$$\sum_{n=0}^{p} \nu^{(p)}(n; L)\frac{\mathrm{d}}{\mathrm{d}r}\rho_{n;L}(r) = 0 \tag{45}$$

in addition to the orthonormality constraint equation

$$\sum_{n=0}^{p} \nu^{(p)*}(n; L)\nu^{(p')}(n; L) = \delta_{p,p'} \tag{46}$$

for any integer $p' > 0$. It can be deduced from the discussion in equations (40)–(43) that the above-defined set of functions $\{u_{p,L,M;L}(\boldsymbol{r})\}$ forms an orthonormal basis for any well-behaved source. For instance, we can expand any well-behaved source as

$$\rho(\boldsymbol{r}) = \sum_{p=0}^{\infty}\sum_{L=0}^{\infty}\sum_{M=-L}^{L} b(p, L, M; L)u_{p,L,M;L}(\boldsymbol{r}) \tag{47}$$

where the expansion coefficients $b(p, L, M; L) = (u_{p,L,M;L}, \rho)_X$. It is important to show how the constraint equations (45), (46) can be jointly satisfied. That this is the case follows from the fact that each basis function $u_{p,L,M;L}(\boldsymbol{r})$ consists of a sum of $p+1$ linearly independent functions, while condition (46) involves only the first $p_0 + 1$ of these functions, where p_0 is the smallest of p and p'. With this clarification, we arrive at the following procedure to construct the orthonormal set. Members $u_{1,L,M;L}(\boldsymbol{r})$ of the set are constructed with $\nu^{(1)}(0; L)$ and $\nu^{(1)}(1; L)$ selected so as to satisfy equations (45), (46) with $p = p' = 1$. Members $u_{2,L,M;L}(\boldsymbol{r})$ of the set are constructed with $\nu^{(2)}(0; L)$ and $\nu^{(2)}(1; L)$ selected so as to obey equation (46) with $p = 1$ and $p' = 2$. This leaves $\nu^{(2)}(2; L)$ arbitrary, and also leaves $\nu^{(2)}(0; L)$ and $\nu^{(2)}(1; L)$ arbitrary up to a multiplicative factor. The multiplicative factor and $\nu^{(2)}(2; L)$ are then uniquely determined from equations (45), (46) with $p = p' = 2$. The general result follows by induction. This approach is illustrated in section 4 for the special case of a spherically symmetric source.

Clearly, equations (44)–(47) enable one to formulate the ISP for well-behaved sources by means of a procedure similar to that employed earlier for $L_2^{(0)}(V)$ sources. In particular, the problem reduces to using the series expansion equation (47) with $u_{p,L,M;L}(\boldsymbol{r})$ defined by equations (44)–(46) and the associated discussion, in place of its $L_2^{(0)}(V)$ analogue, defined by equations (16), (20)–(22). The relevant expansion coefficients and functions $b(p, L, M; L)$ and $u_{p,L,M;L}(\boldsymbol{r})$ thus play the role previously assigned to $a(n, L, M; L)$ and $\rho_{n,L,M;L}(\boldsymbol{r})$, respectively. The remaining steps of the associated source-inversion procedure are developed in section 4 for the special case of a spherically symmetric source. We will then also compare the spherically symmetric case results corresponding to the (three) ISP formulations presented above, corresponding, respectively, to general $L_2(V)$, $L_2^{(0)}(V)$ and well-behaved sources.

4. Special case: spherically symmetric source

The results of section 2, applicable to general $L_2(V)$ sources, and the results of section 3, applicable to $L_2^{(0)}(V)$ and well-behaved sources, are illustrated next for a spherically symmetric source. In this case all the multipole moments of the field, except $g_{0,0}$, vanish. We present first the corresponding analytical results based on sections 2 and 3 above. At the end of the section, we highlight some of our results with the aid of plots for the different cases.

4.1. The $L_2(V)$ case

The normal solution $\hat{\rho}$ to the associated ISP for general $L_2(V)$ sources, without regularity constraints, is defined by equations (11), (15) with the data vector $\boldsymbol{g}$ having trivial multipole moment entries except $g_{0,0}$. One then obtains

$$\hat{\rho}(r) = \sqrt{4\pi}\, g_{0,0}H(a-r)j_0(kr)/\sigma_0^2 \tag{48}$$

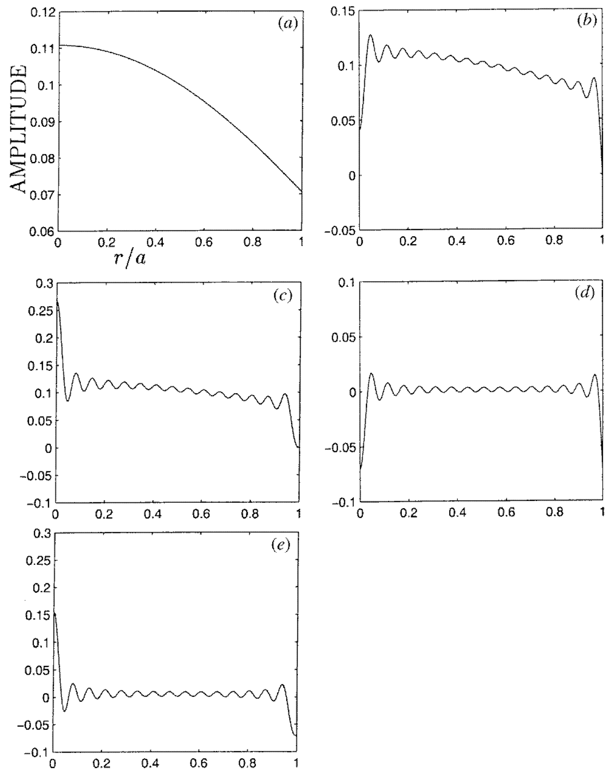

Figure 1. Normal solutions and NR source components, versus r/a, for $ka = \pi/2$: (a) minimum energy solution $\hat{\rho}(r)$. (b) Normal solution $\hat{\rho}^{(0)}(r)$. (c) Normal solution $\hat{\rho}^{(1)}(r)$. (d) NR part $\hat{\rho}_{NR}^{(0)}(r)$ of $\hat{\rho}^{(0)}(r)$. (e) NR part $\hat{\rho}_{NR}^{(1)}(r)$ of $\hat{\rho}^{(1)}(r)$.

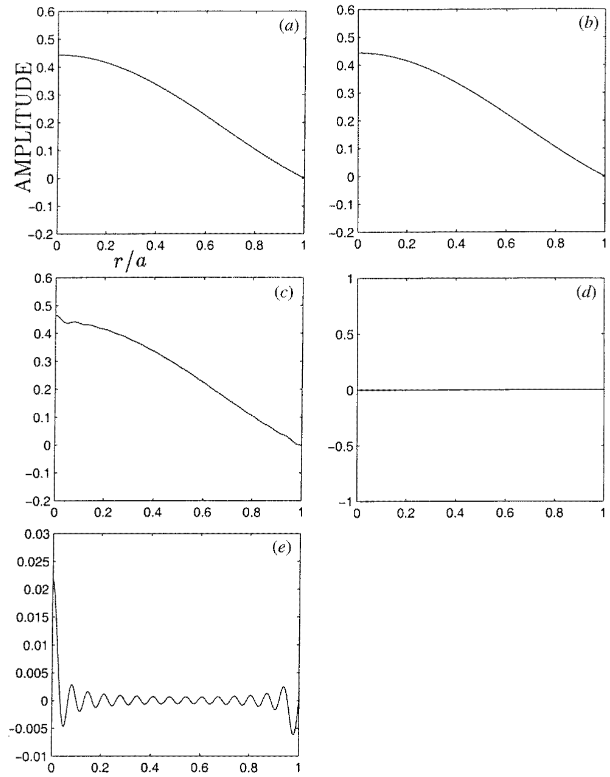

Figure 2. Normal solutions and NR source components, versus r/a, for $ka = \pi$: (a) minimum energy solution $\hat{\rho}(r)$. (b) Normal solution $\hat{\rho}^{(0)}(r)$. (c) Normal solution $\hat{\rho}^{(1)}(r)$. (d) NR part $\hat{\rho}_{NR}^{(0)}(r)$ of $\hat{\rho}^{(0)}(r)$. (e) NR part $\hat{\rho}_{NR}^{(1)}(r)$ of $\hat{\rho}^{(1)}(r)$.

where

$$\sigma_0^2 = 8\pi^2 a k^{-2}[1 - \mathrm{sinc}(2ka)], \tag{49}$$

where we have used $Y_{0,0} = 1/\sqrt{4\pi}$ (see [25], p 682). In deriving equation (49) we have used $j_1(ka) = [\mathrm{sinc}(ka) - \cos ka]/(ka)$ and the recurrence relations of the spherical Bessel functions (see [25], pp 626–7).

4.2. The $L_2^{(0)}(V)$ case

We consider next the corresponding normal solution $\hat{\rho}^{(0)}$ to the ISP for $L_2^{(0)}(V)$ sources addressed in section 3. The normal solution $\hat{\rho}^{(0)}$ associated with a data vector $\boldsymbol{g}$ having trivial multipole moment entries except $g_{0,0}$ is found from equations (29), (36) to be given by equations (48), (49) if ka is a zero of the spherical Bessel function $j_0(\cdot)$ (i.e. $\hat{\rho}$ and $\hat{\rho}^{(0)}$ are then identical). The normal solution corresponding to the general case when ka is not a zero of $j_0(\cdot)$ is described by equations (28), (36) and can be expressed as

$$\hat{\rho}^{(0)}(r) = \frac{1}{\sqrt{4\pi}} g_{0,0} \left[\sum_{n'=0}^{\infty} \alpha_{0,n'}^2 \right]^{-1} \sum_{n=0}^{\infty} \alpha_{0,n} \rho_{n;0}(r) \tag{50}$$

where

$$\begin{aligned} \alpha_{0,n} &= \frac{2\sqrt{2}\pi^2 a^{-1} k^{-1/2} (\beta_{0,n})^{3/2}}{[k^2 - (\beta_{0,n}/a)^2]} J_{1/2}(ka) J'_{1/2}(\beta_{0,n}) \\ &= 4\pi^2\sqrt{2}(-1)^{n+1}(n+1)a^{1/2}k^{-1} \\ &\quad \times \sin(ka)/[(ka)^2 - \beta_{0,n}^2] \end{aligned} \tag{51}$$

and

$$\rho_{n;0}(r) = \sqrt{2/a^3}\beta_{0,n} H(a-r) j_0(\beta_{0,n} r/a) \tag{52}$$

where

$$\begin{aligned} j_0(\beta_{0,n} r/a) &= \frac{\sin(\beta_{0,n} r/a)}{\beta_{0,n} r/a} = \frac{\sin[(n+1)\pi r/a]}{(n+1)\pi r/a} \\ \beta_{0,n} &= (n+1)\pi = |j_1(\beta_{0,n})|^{-1}. \end{aligned} \tag{53}$$

4.3. Well-behaved source case

Finally, we consider the ISP for well-behaved sources. In this case, expressions (44)–(47) (with $L = 0$ and $M = 0$) reduce to

$$u_{p,0,0;0}(r) = \frac{1}{\sqrt{4\pi}} \sum_{n=0}^{p} \nu^{(p)}(n;0) \rho_{n;0}(r), \tag{54}$$

$$\sum_{n=0}^{p} (-1)^{n+1}(n+1)\nu^{(p)}(n;0) = 0, \tag{55}$$

$$\sum_{n=0}^{p} \nu^{(p)*}(n;0)\nu^{(p')}(n;0) = \delta_{p,p'} \tag{56}$$

and

$$\rho(r) = \sum_{p=0}^{\infty} b(p,0,0;0) u_{p,0,0;0}(r). \tag{57}$$

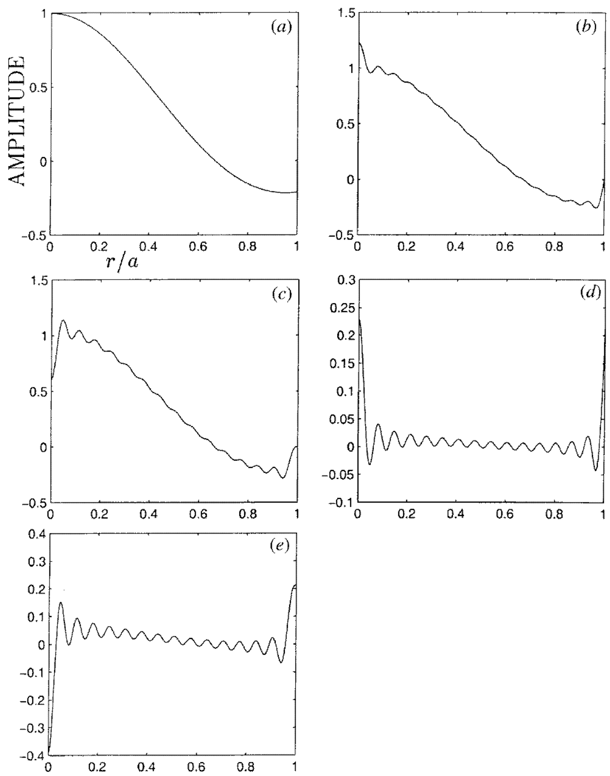

Figure 3. Normal solutions and NR source components, versus r/a, for $ka = 1.5\pi$: (a) minimum energy solution $\hat{\rho}(r)$. (b) Normal solution $\hat{\rho}^{(0)}(r)$. (c) Normal solution $\hat{\rho}^{(1)}(r)$. (d) NR part $\hat{\rho}^{(0)}_{NR}(r)$ of $\hat{\rho}^{(0)}(r)$. (e) NR part $\hat{\rho}^{(1)}_{NR}(r)$ of $\hat{\rho}^{(1)}(r)$.

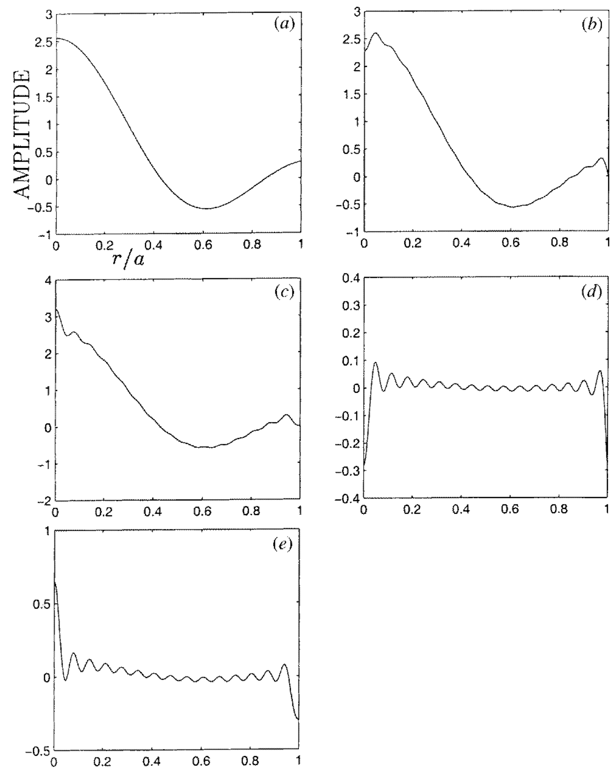

Figure 4. Normal solutions and NR source components, versus r/a, for $ka = 2.33\pi$: (a) minimum energy solution $\hat{\rho}(r)$. (b) Normal solution $\hat{\rho}^{(0)}(r)$. (c) Normal solution $\hat{\rho}^{(1)}(r)$. (d) NR part $\hat{\rho}^{(0)}_{NR}(r)$ of $\hat{\rho}^{(0)}(r)$. (e) NR part $\hat{\rho}^{(1)}_{NR}(r)$ of $\hat{\rho}^{(1)}(r)$.

The coefficients $\nu^{(p)}(n; 0)$ in equations (54)–(56) are defined by

$$v_j^{(p)}(0) = \left\{ \sum_{n=0}^{p-1}(n+1)^2 + \frac{[\sum_{n=0}^{p-1}(n+1)^2]^2}{(p+1)^2} \right\}^{-1/2}, \quad (58)$$

$$v_j^{(p)}(n) = (-1)^n (n+1) v_j^{(p)}(0) \quad 0 < n < p \quad (59)$$

and

$$v_j^{(p)}(p) = v_j^{(p)}(0) \frac{(-1)^{p+1} \sum_{n=0}^{p-1}(n+1)^2}{p+1}. \quad (60)$$

With these results, the normal solution $\hat{\rho}^{(1)}$ to the associated ISP can be derived by means of a procedure similar to that employed above for $L_2^{(0)}(V)$ sources. We obtain

$$\hat{\rho}^{(1)}(r) = g_{0,0} \left[\sum_{p'=0}^{\infty} \gamma_{p'}^2 \right]^{-1} \sum_{p=0}^{\infty} \gamma_p u_{p,0,0;0}(r) \quad (61)$$

where

$$\gamma_p = \sum_{n=0}^{p} \nu^{(p)}(n; 0) \alpha_{0,n} \quad (62)$$

with $\alpha_{0,n}$ defined by equations (28), (29) (note the similarity between equations (61), (62) and their $L_2^{(0)}(V)$ counterparts, equations (50), (51)).

4.4. Numerical illustration: $L_2(V)$, $L_2^{(0)}(V)$ and well-behaved source cases

In the following plots we have normalized the normal solutions with respect to $g_{0,0}/a^3$. Figures 1–5 show plots of the normal solutions defined above, versus r/a, corresponding to the $L_2(V)$, $L_2^{(0)}(V)$ and well-behaved source cases, for different values of the normalized wavenumber ka. Also shown are plots of the corresponding NR parts $\hat{\rho}^{(0)}_{NR}$ and $\hat{\rho}^{(1)}_{NR}$.

We note that, in contrast to the general $L_2(V)$ case, for $L_2^{(0)}(V)$ sources (i.e., with the additional compact supportness constraint) the normal solution to the ISP is guaranteed to vanish on the boundary ∂V of the source volume V. This holds regardless of the value of ka. The plots corresponding to the normal solutions to ISPs with and without the compact supportness constraint coincide only if ka is a zero of $j_0(\cdot)$, i.e., for $ka = (n+1)\pi$, where n is an integer. This is to be expected since, in the latter case, the normal solution defined by equations (48), (49) possesses compact support in V. For the well-behaved source case, the associated normal solutions possess (additionally) a continuous normal derivative on the boundary of the source region V. We see that the NR parts $\hat{\rho}^{(0)}_{NR}$ and $\hat{\rho}^{(1)}_{NR}$ are, in general, nontrivial. The nontrivial NR source components corresponding to the well-behaved source case are clearly more visible than those for the $L_2^{(0)}(V)$ case.

These results are consistent with results derived recently in [17]. In particular, it was shown in [17] that in order for a localized source (in this case, a source to the inhomogeneous Helmholtz equation) to lack a NR part, it must necessarily obey the homogeneous form of the corresponding partial differential equation (e.g. the Helmholtz equation) in the interior of its support. This automatically explains why,

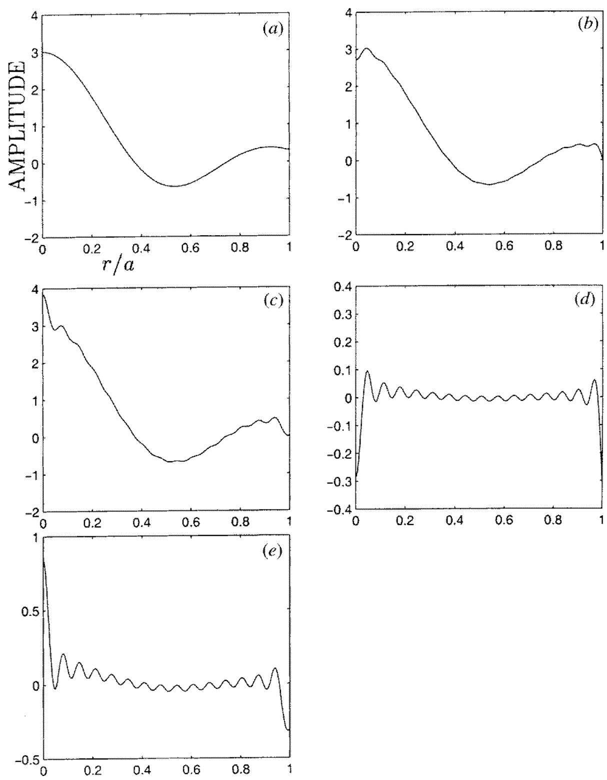

Figure 5. Normal solutions and NR source components, versus r/a, for $ka = 2.67\pi$: (*a*) minimum energy solution $\hat{\rho}(r)$. (*b*) Normal solution $\hat{\rho}^{(0)}(r)$. (*c*) Normal solution $\hat{\rho}^{(1)}(r)$. (*d*) NR part $\hat{\rho}^{(0)}_{NR}(r)$ of $\hat{\rho}^{(0)}(r)$. (*e*) NR part $\hat{\rho}^{(1)}_{NR}(r)$ of $\hat{\rho}^{(1)}(r)$.

out of all $L_2^{(0)}(V)$ sources, only those $L_2^{(0)}(V)$ sources that are also resonant wave solutions lack a NR part. We verify this situation in figure 2. This also explains why minimum energy solutions are homogeneous wave solutions (see expression (15) in section 2 and its spherically symmetric version equation (48) in this section). Now, it is not hard to show that no source that vanishes along with its normal derivatives on the boundary of a given spherical domain can obey the requirement of being a homogeneous wave solution (in particular, no zero of the spherical Bessel function $j_l(\cdot)$ is also a zero of $j_l'(\cdot)$). One arrives at the same conclusion for a more general source, confined in an arbitrary simply connected source region, by noting that the only solution to the homogeneous Helmholtz equation which obeys the above-imposed overspecified boundary conditions is the trivial solution. Thus, no source exists that is both well behaved and lacks a NR part. In the present context, we see that not even normal solutions to ISPs for such well-behaved sources lack a NR part.

5. Conclusion

In this paper, we investigated the ISP for general L_2 sources confined within a given spherical volume. We also investigated two, more restricted versions of the ISP, with additional regularity (smoothness) constraints: an ISP for L_2 sources that possess compact support in a given source region (such sources therefore vanish on the boundary of the specified support), and an ISP for well-behaved sources (L_2 sources that vanish along with their normal derivatives on the boundary of their specified support). Expressions for the normal solutions and their associated NR parts were derived corresponding to the ISP formulations considered. The formalism developed in the paper makes use of standard linear inversion theory in addition to spherical harmonics and Bessel functions and can be applied to other forms of ISP, with other regularity constraints.

For the ISP without regularity constraints, the corresponding normal solution is the usual minimum energy solution. The latter is orthogonal to all L_2 NR sources in the source's support. It thus lacks a NR part. For the ISPs with regularity constraints addressed in this paper the situation is different: the associated normal solutions possess, in general, nontrivial NR parts. From an inversion point of view, we thus established a strategy for extracting NR source components of an unknown source, by imposing *a priori* constraints of regularity, in addition to the usual localization constraint. It is worth emphasising, however, that the normal solutions corresponding to the ISPs with regularity constraints illustrated here had the form of (only) small-perturbation versions of their affiliated minimum energy solutions. Naturally, the associated perturbation was seen to increase as we imposed further regularity constraints.

The present discussion also illustrated some recently derived properties of NR sources and purely radiating sources (i.e. sources that lack a NR part). Our formulation, applicable to a spherical coordinate system, can be generalized to other (separable) systems.

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