Aperture realizations of exact solutions to homogeneous-wave equations

Richard W. Ziolkowski

Electromagnetics Laboratory, Department of Electrical and Computer Engineering, The University of Arizona, Tucson, Arizona 85721

Ioannis M. Besieris

Bradley Department of Electrical Engineering, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

Amr M. Shaarawi

Department of Engineering Physics and Mathematics, Faculty of Engineering, Cairo University, Giza, Egypt

Received January 21, 1992; revised manuscript received July 13, 1992; accepted July 28, 1992

Several new classes of localized solutions to the homogeneous scalar wave and Maxwell's equations have been reported recently. Theoretical and experimental results have now clearly demonstrated that remarkably good approximations to these acoustic and electromagnetic localized-wave solutions can be achieved over extended near-field regions with finite-sized, independently addressable, pulse-driven arrays. We demonstrate that only the forward-propagating (causal) components of any homogeneous solution of the scalar-wave equation are actually recovered from either an infinite- or a finite-sized aperture in an open region. The backward-propagating (acausal) components result in an evanescent-wave superposition that plays no significant role in the radiation process. The exact, complete solution can be achieved only from specifying its values and its derivatives on the boundary of any closed region. By using those localized-wave solutions whose forward-propagating components have been optimized over the associated backward-propagating terms, one can recover the desirable properties of the localized-wave solutions over the extended near-field regions of a finite-sized, independently addressable, pulse-driven array. These results are illustrated with an extreme example—one dealing with the original solution, which is superluminal, and its finite aperture approximation, a slingshot pulse.

1. INTRODUCTION

Large classes of nonseparable space–time solutions of the equations that govern many wave phenomena (e.g., scalar-wave,[1–7] Maxwell's,[2,8] and Klein–Gordon[9] equations) have been reported recently. When compared with traditional monochromatic, continuous-wave (cw) solutions, these localized-wave (LW) solutions are characterized by extended regions of localization; i.e., their shapes and/or amplitudes are maintained over much larger distances than are those of their cw analogs. This is also true in complex environments such as naturally dispersive media (waveguides[3,9,10] and plasmas[11]) and lossy media.[12,13] These discoveries have prompted several extensive investigations into the possibility of using these LW solutions to drive finite-sized arrays, thereby launching fields that have extended localization properties.[14,15] Pulses with these desirable localized transmission characteristics could have a number of potential applications in the areas of directed energy transmission, secure communications, and remote sensing.

For many idealistic situations, the scalar-wave equation is an adequate model of the underlying physics that govern wave propagation and scattering. To simplify the discussion, it will be the only case considered here. It has been shown[3,4] that exact LW solutions of this equation can be obtained from a representation that uses a decomposition into bidirectional-traveling plane-wave solutions, i.e., solutions formed as a product of forward- and backward-traveling plane waves. This bidirectional representation is readily connected to one dealing with only distinct sums of forward- and backward-propagating plane waves. It is complete and invertible. The bidirectional representation does not replace the standard Fourier synthesis but rather complements it, especially for the LW class of solutions.

If the LW solutions could be recovered from a finite aperture or array, the above-mentioned applications might be realized in practice. It is well known, however, that many of the LW solutions are composed of both forward- and backward-propagating wave components.[2,16,17] It has been argued[16,17] that this causes grave problems with causality and with the potential realization of systems that take advantage of these LW solutions. One of our main purposes in this paper is to explain how these solutions have been reproduced with a strictly causal Green's function, at least in an approximate sense in the near field of an aperture (array).

As is demonstrated in Section 2, only the forward-propagating components of any solution of the homogeneous-wave equation are recovered in an open region from an infinite aperture. In analogy with the results of Sherman *et al.*,[18] Devaney and Sherman,[19] and Zharii,[20] it is shown that the remaining backward-propagating components can be represented as a superposition of evanes-

0740-3232/93/010075-13$05.00

cent waves. Explicit comparisons of the contributions from the forward- and the backward-propagating components are made. The finite aperture results pertaining to diffraction lengths, beam spread, etc. have been addressed elsewhere.[21,22] It is also demonstrated that the LW solutions can be designed to minimize the contributions from the backward-propagating components and hence that the causal field generated by driving a finite aperture with one of these LW solutions is a demonstrably close approximation to the original solution everywhere within the diffraction length of the initial aperture. This synthesis of the wave-equation solutions in the near field of a finite aperture is discussed in Section 3. The relationship of the present results to those recently reported by Hillion[23] is given in Section 4. A class of superluminal LW solutions is considered in Section 5 to emphasize that approximations of these LW solutions can be realized in the near field of a finite aperture. The resulting slingshot pulses exhibit the superluminal and enhanced localization characteristics of the infinite aperture solution in that region. The results of the research presented in this paper are summarized in Section 6.

2. RECONSTRUCTION OF HOMOGENEOUS, SCALAR-WAVE-EQUATION SOLUTIONS FROM THE HUYGENS REPRESENTATION

A variety of novel classes of solutions of the homogeneous scalar-wave equation (HWE),

$$\{\Delta - \partial_{ct}^2\} f(\mathbf{r}, t) = 0, \tag{2.1}$$

have recently been under investigation.[1-13] These solutions are characterized by their enhanced localization properties; they have been used to drive arrays that result in beams that share these properties.[14,15]

Consider the Huygens representation of a field $f(\mathbf{r}, t)$ that is forward propagating into the region $z > 0$ from initial data given on the plane $\Sigma = \{(x, y, z) \mid z = 0\}$. The resultant field is given by[24]

$$g(\mathbf{r}, t) = \int_\Sigma dS', \quad \Psi(x', y', z' = 0, t - R/c) \frac{1}{4\pi R}, \tag{2.2a}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$. The term Ψ in Eq. (2.2a) is defined as

$$\Psi(x', y', z', t - R/c) = -[\partial_{z'} f] + [\partial_{ct'} f] \frac{(z - z')}{R} + [f] \frac{(z - z')}{R^2}. \tag{2.2b}$$

All quantities in brackets in Eq. (2.2b) are evaluated at the indicated retarded time. This representation can be considered as an operation $\mathcal{H}_\Sigma$ that takes the projection of the field f and its derivatives on Σ to the function $g = \mathcal{H}_\Sigma[f]$. As we discuss in Appendix A, HWE solutions that are foward propagating when they encounter the sphere at infinity will be uniquely reconstructed by this representation, i.e., $g \equiv f$.

Consider now the Fourier–Bessel representation of an axisymmetric (azimuthally symmetric) LW pulse. It can be readily addressed from the bidirectional representation given in Ref. 3. In particular, it can be expressed in terms of forward- and backward-propagating Bessel beams by the expression[2]

$$f(\mathbf{r}, t) = \int_0^\infty \chi d\chi \int_{-\infty}^\infty dk_z \int_{-\infty}^\infty d\omega A_0(\chi, k_z, \omega) J_0(\chi\rho) \times \exp[-i(k_z z - \omega t)] \delta(\omega^2 - [\chi^2 + k_z^2] c^2). \tag{2.3}$$

The bidirectional spectrum $C_0(\chi, \alpha, \beta)$ is related to this Fourier spectrum as[3]

$$A_0(\chi, k_z, \omega) = 2c C_0\left[\chi, \frac{1}{2}\left(\frac{\omega}{c} + k_z\right), \frac{1}{2}\left(\frac{\omega}{c} - k_z\right)\right]. \tag{2.4}$$

Accounting for the delta function constraint in terms of the variable ω, we can rewrite Eq. (2.3) as

$$\begin{aligned} f(\mathbf{r}, t) = &\int_0^\infty d\chi\chi J_0(\chi\rho) \\ &\times \int_0^\infty dk_z \left\{ A_0(\chi, k_z, \omega_+) \frac{1}{2\omega_+} \exp[-i(k_z z - \omega_+ t)] \right. \\ &\left. + A_0(\chi, -k_z, \omega_+) \frac{1}{2\omega_+} \exp[+i(k_z z + \omega_+ t)] \right\} \\ &+ \int_0^\infty d\chi\chi J_0(\chi\rho) \\ &\times \int_0^\infty dk_z \left\{ A_0(\chi, k_z, -\omega_+) \frac{1}{2\omega_+} \exp[-i(k_z z + \omega_+ t)] \right. \\ &\left. + A_0(\chi, -k_z, -\omega_+) \frac{1}{2\omega_+} \exp[+i(k_z z - \omega_+ t)] \right\}, \end{aligned} \tag{2.5}$$

where $\omega_+ = +(\chi^2 + k_z^2)^{1/2} c$. If we assume that $f(\mathbf{r}, t)$ is real, the spectrum $A_0(\chi, k_z, \omega)$ must satisfy the conjugation property (crossing relationship)

$$A_0(\chi, k_z, \omega) = A_0^*(\chi, -k_z, -\omega). \tag{2.6}$$

As a consequence, one can write Eq. (2.5) as

$$\begin{aligned} f(\mathbf{r}, t) = &\operatorname{Re} \int_0^\infty d\chi\chi J_0(\chi\rho) \int_0^\infty dk_z \frac{A_0[\chi, k_z, (\chi^2 + k_z^2)^{1/2} c]}{(\chi^2 + k_z^2)^{1/2} c} \\ &\times \exp\{-i[k_z z - (\chi^2 + k_z^2)^{1/2}(ct)]\} \\ &+ \operatorname{Re} \int_0^\infty d\chi\chi J_0(\chi\rho) \int_0^\infty dk_z \frac{A_0[\chi, -k_z, (\chi^2 + k_z^2)^{1/2} c]}{(\chi^2 + k_z^2)^{1/2} c} \\ &\times \exp\{+i[k_z z + (\chi^2 + k_z^2)^{1/2}(ct)]\} \\ \equiv &f_+(\mathbf{r}, t) + f_-(\mathbf{r}, t), \end{aligned} \tag{2.7}$$

which clearly isolates the forward f_+ and the backward f_- propagating Bessel beam components.

With respect to the possibility of physically generating such solutions from a realistic source configuration, it is of major importance to understand how the Huygens operator $\mathcal{H}_\Sigma$ acts on the forward- and the backward-propagating Bessel beam components of these HWE solutions. This behavior is derived explicitly in Appendix B. One finds that

$$g = \mathcal{H}_\Sigma[f] = \mathcal{H}_\Sigma[f_+] + \mathcal{H}_\Sigma[f_-] \equiv f_+, \tag{2.8}$$

since

$$\mathcal{H}_\Sigma[J_0(\chi\rho)\exp(-i(k_z z - \omega t))] \equiv J_0(\chi\rho)\exp[-i(k_z z - \omega t)], \tag{2.9a}$$

$$\mathcal{H}_\Sigma[J_0(\chi\rho)\exp(-i(k_z z - \omega t))] \equiv 0. \tag{2.9b}$$

This means that the Huygens operator filters out any backward-propagating components, i.e., the backward-propagating components are superfluous to launching a wave into the region $z > 0$. Thus only the forward-propagating component of the LW is reconstructed from the initial data upon an open surface by the causal propagator. If, on the other hand, the observation point were in the region $z < 0$, then the results of Appendix B show that only the backward-propagating components are recovered by the causal Huygens operator. Thus the exact solution would be reconstructed in a closed region bounded by two infinite planes. One surface would generate the forward-propagating components; the other would generate the backward-propagating components.

The presence of backward-propagating components in the exact LW solutions has led to various statements that suggest "grave difficulties"[16,17] with the potential physical realization of these solutions. However, as is shown in Refs. 2 and 3, the bidirectional spectra may be chosen to make the amount of the backward-propagating components in the resulting solutions negligible. This "tweaking up" of the spectra to obtain a forward-propagating beam was confirmed in Ref. 7. The result above [Eq. (2.8)] shows that, if the HWE solution had large backward-propagating components, it would not be reconstructed well from the initial boundary data, since a large portion of the field would be lost. On the other hand, this result also shows that, if the HWE solution contained a negligible amount of backward-propagating components, the reconstructed function would not be corrupted, and a forward-propagating beam would result that would exhibit the localization properties of the original HWE solution.

To quantify this ability to design the spectra to one's advantage, we isolate the forward- and the backward-propagating components and then compare their intensities. We accomplish this isolation process by reinterpreting the meaning of the backward-propagating components in terms of evanescent waves in order to make a clear identification of their contributions. We find that the forward-propagating components are emphasized at the expense of the backward-propagating ones when we increase the frequency bandwidth of the signals.

In contrast to the process that led from Eq. (2.3) to Eq. (2.7), let us consider the delta function constraint as a function of k_z instead of ω. Since it requires that $k_z = [(\omega/c)^2 - \chi^2]^{1/2}$, we make the following choice for the branch cut:

$$k_z = \begin{cases} [(\omega/c)^2 - \chi^2]^{1/2} & \text{for } \omega > \chi c \\ -i[\chi^2 - (\omega/c)^2]^{1/2} & \text{for } |\omega| < \chi c \\ -[(\omega/c)^2 - \chi^2]^{1/2} & \text{for } \omega < -\chi c \end{cases} \cdot \tag{2.10}$$

Equation (2.3) then yields

$$\begin{aligned} f(\mathbf{r},t) &= \int_0^\infty \mathrm{d}\chi\chi J_0(\chi\rho)\Bigg\{\int_{\chi c}^\infty \mathrm{d}\omega\Bigg[\frac{A_0(\chi,[(\omega/c)^2-\chi^2]^{1/2},\omega)}{|(\omega/c)^2-\chi^2|^{1/2}c^2}\exp(-i\{[(\omega/c)^2-\chi^2]^{1/2}z-\omega t\}) \\ &\quad + \frac{A_0(\chi,-[(\omega/c)^2-\chi^2]^{1/2},-\omega)}{|(\omega/c)^2-\chi^2|^{1/2}c^2}\exp(+i\{[(\omega/c)^2-\chi^2]^{1/2}z-\omega t\})\Bigg] \\ &\quad + \int_0^{\chi c}\mathrm{d}\omega\Bigg[\frac{A_0(\chi,-i[\chi^2-(\omega/c)^2]^{1/2},\omega)}{|\chi^2-(\omega/c)^2|^{1/2}c^2}\exp(+i\omega t)\exp\{-[\chi^2-(\omega/c)^2]^{1/2}z\} \\ &\quad + \frac{A_0(\chi,i[\chi^2-(\omega/c)^2]^{1/2},-\omega)}{|\chi^2-(\omega/c)^2|^{1/2}c^2}\exp(-i\omega t)\exp\{-[\chi^2-(\omega/c)^2]^{1/2}z\}\Bigg]\Bigg\} \\ &= \mathrm{Re}\int_0^\infty \mathrm{d}\chi\chi J_0(\chi\rho)\int_{\chi c}^\infty \mathrm{d}\omega\frac{A_0(\chi,[(\omega/c)^2-\chi^2]^{1/2},\omega)}{|(\omega/c)^2-\chi^2|^{1/2}c^2}\exp(-i\{[(\omega/c)^2-\chi^2]^{1/2}z-\omega t\}) \\ &\quad + \mathrm{Re}\int_0^\infty \mathrm{d}\chi\chi J_0(\chi\rho)\int_{\infty/0}^{\chi c}\mathrm{d}\omega\frac{A_0(\chi,-i[\chi^2-(\omega/c)^2]^{1/2},\omega)}{|\chi^2-(\omega/c)^2|^{1/2}c^2}\exp\text{-}(+i\omega t)\exp\{-[\chi^2-(\omega/c)^2]^{1/2}z\}. \end{aligned} \tag{2.11}$$

Thus one obtains a representation in terms of a superposition of forward-propagating Bessel beams and a superposition over waves that are evanescent in the z direction. Such a decomposition is expected from previous research on representations of scalar-wave fields.[18–20]

To give an illustrative example of the formalisms above, we consider the azimuthally symmetric focus wave mode (FWM) pulse[1–3]:

$$\begin{aligned} f_{\mathrm{FWM}}(\rho, z-ct, z+ct) &= \frac{z_0}{z_0 + i(z-ct)}\exp[ik(z+ct)] \\ &\quad \times \exp\{-k\rho^2/[z_0+i(z-ct)]\}. \end{aligned} \tag{2.12}$$

The real constants k and z_0 are free parameters; their meaning is revealed when we take the Fourier transform of this FWM pulse. As is shown in Ref. 13 or equivalently in Appendix C, one obtains

$$\begin{aligned} \hat{f}_{\mathrm{FWM}}(\rho, z, \omega) &= \int_{-\infty}^\infty \mathrm{d}t\,\exp(-i\omega t) f_{\mathrm{FWM}}(\rho, z-ct, z+ct) \\ &= \begin{cases} \dfrac{2\pi z_0}{c}\exp[k(z_0+2iz)]\exp[-\omega(z_0+iz)/c]J_0(2\rho\{k[(\omega/c)-k]\}^{1/2}) & \text{for } \omega > kc \\ 0 & \text{for } \omega < kc \end{cases} \cdot \end{aligned} \tag{2.13}$$

The indicated transform kernel is chosen to match bidirectional-plane-wave representation (2.3), which emphasizes the $\exp(+i\omega t)$ inverse kernel. As $\mathcal{F}_t\{f_{\mathrm{FWM}}{}^*\}(\rho, z, \omega) = \mathcal{F}_t\{f_{\mathrm{FWM}}\}(\rho, z, -\omega)$, where $\mathcal{F}_t\{f_{\mathrm{FWM}}\} \equiv \hat{f}_{\mathrm{FWM}}$, we see that the Fourier transform $\mathcal{F}_t\{\mathrm{Re}[f_{FWM}]\}$ will contain no temporal frequencies in the range $(-kc, +kc)$. The $\exp[ik(z + ct)]$ term in the FWM pulse equation (2.12) thus acts as a high-pass filter that permits only those frequencies $\omega > kc$. The spectrum $\hat{f}_{\mathrm{FWM}}(\rho, z, \omega)$ has an exponential roll-off, the $1/e$ point being reached at $\omega_{\max} = c/z_0$. Note that the waist of the FWM pulse in the plane $z = 0$ is given by $w_0{}^2 = z_0/k$. Therefore the parameter that represents the inverse of the asymptotic term kz_0 is

$$\Lambda = \frac{1}{kz_0} = \left(\frac{w_0}{z_0}\right)^2 = \left(\frac{2\pi w_0}{\lambda_{\min}}\right)^2 = \frac{\omega_{\max}}{\omega_{\min}}. \qquad (2.14)$$

The parameter Λ thus characterizes the frequency bandwidth required for the FWM pulse or, equivalently, the square of the number of minimum wavelengths ($\lambda_{\min} = 2\pi c/\omega_{\max} = 2\pi z_0$) in the circumference of the circle surrounding its initial waist. This Fourier spectrum [Eq. (2.13)] shows that the asymptotic limit investigated by Heyman *et al.*,[16] where $kz_0 \gg 1$, is a nonsensical regime for the FWM pulse in that it requires that $\Lambda < 1$ and hence that $\omega_{\max} < \omega_{\min}$. The backward-propagating dominance of the wave propagation in that limit is to be expected, since one is turning the spectrum inside out. On the other hand, as the bandwidth increases so that Λ increases, the number of minimum wavelengths within the waist increases. From the calculations presented below, we know that this means that the amount of the backward-propagating components decreases.

In particular, the FWM pulse can be written in the forward- and the backward-propagating form [Eq. (2.7)]:

$$\begin{aligned} f_{\mathrm{FWM}}(\rho, z - ct, z + ct) &= f_+^{\mathrm{FWM}}(\rho, z, t) + f_-^{\mathrm{FWM}}(\rho, z, t) \\ &= \int_0^\infty \mathrm{d}\chi\chi J_0(\chi\rho)[\hat{f}_+^{\mathrm{FWM}}(\chi, z, t) + \hat{f}_-^{\mathrm{FWM}}(\chi, z, t)], \end{aligned} \qquad (2.15)$$

where

$$\begin{aligned} \hat{f}_+^{\mathrm{FWM}}(\chi, z, t) &= \int_0^\infty \mathrm{d}k_z \frac{\pi}{(\chi^2 + k_z{}^2)^{1/2}} \exp\{-(z_0/2)[(\chi^2 + k_z{}^2)^{1/2} + k_z]\} \\ &\quad \times \delta[(\chi^2 + k_z{}^2)^{1/2} - k_z - 2k] \\ &\quad \times \exp\{-i[k_z z - (\chi^2 + k_z{}^2)^{1/2}ct]\} \\ &= \frac{\pi}{2k} \exp[ik(z + ct)] \\ &\quad \times \exp\{-(\chi^2/4k)[z_0 + i(z - ct)]\}H(\chi - 2k), \end{aligned} \qquad (2.16a)$$

$$\begin{aligned} \hat{f}_-^{\mathrm{FWM}}(\chi, z, t) &= \int_0^\infty \mathrm{d}k_z \frac{\pi}{(\chi^2 + k_z{}^2)^{1/2}} \exp\{-(z_0/2)[(\chi^2 + k_z{}^2)^{1/2} + k_z]\} \\ &\quad \times \delta[(\chi^2 + k_z{}^2)^{1/2} + k_z - 2k] \\ &\quad \times \exp\{+i[k_z z + (\chi^2 + k_z{}^2)^{1/2}ct]\} \\ &= \frac{\pi}{2k} \exp[ik(z + ct)] \\ &\quad \times \exp\{-(\chi^2/4k)[z_0 + i(z - ct)]\}H(2k - \chi), \end{aligned} \qquad (2.16b)$$

where $H(x)$ is the Heaviside function. On the other hand, the representation that corresponds to Eq. (2.11) yields the same result:

$$\begin{aligned} \hat{f}_+^{\mathrm{FWM}}(\chi, z, t) &= \int_{\chi c}^\infty \mathrm{d}\omega \frac{\pi}{c[(\omega/c)^2 - \chi^2]^{1/2}} \\ &\quad \times \exp(-(z_0/2)\{(\omega/c) + [(\omega/c)^2 - \chi^2]^{1/2}\}) \\ &\quad \times \delta\left\{\frac{\omega}{c} + [(\omega/c)^2 - \chi^2]^{1/2} - 2k\right\} \\ &\quad \times \exp(-i\{[(\omega/c)^2 - \chi^2]^{1/2}z - \omega t\}) \\ &= \frac{\pi}{2k} \exp[ik(z + ct)] \\ &\quad \times \exp\{-(\chi^2/4k)[z_0 + i(z - ct)]\}H(\chi - 2k), \end{aligned} \qquad (2.17a)$$

$$\begin{aligned} \hat{f}_-^{\mathrm{FWM}}(\chi, z, t) &= \int_0^{\chi c} \mathrm{d}\omega \frac{\pi i}{c[\chi^2 - (\omega/c)^2]^{1/2}} \\ &\quad \times \exp(-(z_0/2)\{(\omega/c) - i[\chi^2 - (\omega/c)^2]^{1/2}\}) \\ &\quad \times \delta\left\{\frac{\omega}{c} + i[\chi^2 - (\omega/c)^2]^{1/2} - 2k\right\} \\ &\quad \times \exp(+i\omega t)\exp\{-[\chi^2 - (\omega/c)^2]^{1/2}z\} \\ &= \frac{\pi}{2k} \exp[ik(z + ct)] \\ &\quad \times \exp\{-(\chi^2/4k)[z_0 + i(z - ct)]\}H(2k - \chi). \end{aligned} \qquad (2.17b)$$

Thus we have confirmed that the superposition that corresponds to the backward-propagating waves is equivalent to the superposition that corresponds to the evanescent waves.

The relative contributions of f_+^{FWM} and f_-^{FWM} to the FWM pulse can be readily assessed from their peak values along the axis of propagation. Consider the integrals

$$\begin{aligned} |f_+^{\mathrm{FWM}}(0, z - ct, z + ct)| &= \left|\int_0^\infty \mathrm{d}\chi\chi\hat{f}_+^{\mathrm{FWM}}(\chi, z, t)\right| \\ &\le \frac{2\pi}{kz_0{}^2} \int_{kz_0}^\infty \mathrm{d}\tilde{\chi}\tilde{\chi} \exp[-\tilde{\chi}^2/(kz_0)] \\ &= \frac{\pi}{z_0} \exp(-kz_0), \end{aligned} \qquad (2.18a)$$

$$\begin{aligned} |f_-^{\mathrm{FWM}}(0, z - ct, z + ct)| &= \left|\int_0^\infty \mathrm{d}\chi\chi\hat{f}_-^{\mathrm{FWM}}(\chi, z, t)\right| \\ &\le \frac{2\pi}{kz_0{}^2} \int_0^{kz_0} \mathrm{d}\tilde{\chi}\tilde{\chi} \exp[-\tilde{\chi}^2/(kz_0)] \\ &= \frac{\pi}{z_0}[1 - \exp(-kz_0)]. \end{aligned} \qquad (2.18b)$$

Note that a change of variables $\tilde{\chi} = (z_0/2)\chi$ was used. Therefore the ratio between the peak intensities associated with the forward- and the backward-propagating components is

$$\frac{|f_+^{\mathrm{FWM}}|^2}{|f_-^{\mathrm{FWM}}|^2} \sim \left|\frac{\exp(-kz_0)}{1 - \exp(-kz_0)}\right|^2. \qquad (2.19)$$

This means that for $\Lambda \gg 1$ we have

$$\frac{|f_+^{\mathrm{FWM}}|^2}{|f_-^{\mathrm{FWM}}|^2} \sim \left|\frac{1-(1/\Lambda)}{1/\Lambda}\right|^2 \sim \Lambda^2. \tag{2.20}$$

Consequently, as the bandwidth increases, the forward-propagating components dominate the backward-propagating ones. In fact, for a modest value of $\Lambda = 10$, intensity ratio (2.19) is 90.41, which means that very little (~1.1%) of the FWM pulse will be lost if the infinite plane is driven with such a solution.

The phase velocities along the z axis of the forward- and the backward-propagating component waves are readily extracted from the χ representations [Eqs. (2.15) and (2.16)]. In particular,

$$v_p = \frac{\omega}{k_z} = \frac{(\chi^2 + 4k^2)c/4k}{(\chi^2 - 4k^2)/4k} = \left(\frac{\chi^2 + 4k^2}{\chi^2 - 4k^2}\right)c. \tag{2.21}$$

For the forward- (backward-) propagating terms, the constraints require that $\chi > 2k$ ($\chi < 2k$) so that the phase velocity is positive (negative), with magnitude greater than c. At the boundary where $\chi = 2k$ the phase velocity is infinite, corresponding to a stationary oscillation. On the other hand, the group velocity along the z axis in either case is

$$v_g = \frac{\partial\omega}{\partial k_z} = \frac{\partial\omega}{\partial\chi}\frac{\partial\chi}{\partial k_z} = \frac{\chi}{2k}c \times \frac{2k}{\chi} = +c. \tag{2.22}$$

These results further confirm the interpretation of the forward-propagating components as being the modes that radiate energy away from the source plane and the backward propagating modes as being the evanescent modes, i.e., the modes that store energy near the source plane. Analogous results are known for simple pulsed sources (see, for instance, Ref. 25, p. 493).

3. FINITE-APERTURE RECONSTRUCTION OF HOMOGENEOUS SCALAR-WAVE-EQUATION SOLUTIONS

The analysis given in Section 2 is restricted here to the case of a finite aperture. If we use standard arguments[21] and assume that the observation point is sufficiently far away from the aperture that for any forward-propagating components $[\partial_z f] \sim -[\partial_{ct} f]$, a reasonably good approximation of the field generated by driving a circular aperture $\mathcal{A}$ of radius a in the plane $z = 0$ with the HWE solution $f(\mathbf{r}, t)$ is

$$g(\mathbf{r}, t) \approx \int_{\mathcal{A}} \mathrm{d}S' \frac{2[\partial_{ct'} f(x', y', z' = 0, t')](t' = t - R/c)}{4\pi R}. \tag{3.1}$$

Thus, if the driving function is a Bessel beam $f(\mathbf{r}, t) = J_0(\chi\rho)\exp[-i(k_z z - \omega t)]$, then one has simply

$$g(\mathbf{r}, t) \approx \left(\frac{+2i\omega}{c}\right)\int_0^{2\pi} \mathrm{d}\phi' \int_0^a \mathrm{d}\rho'\rho' \frac{J_0(\chi\rho')}{4\pi R} \times \exp[+i\omega(t - R/c)]. \tag{3.2}$$

Since for $z \gg \rho$, $z \gg \rho'$,

$$R = [\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + z^2]^{1/2} \sim z + \frac{\rho^2 + \rho'^2}{2z} - \frac{\rho\rho'}{z}\cos(\phi - \phi'), \tag{3.3}$$

one has, approximately, with the standard integral representation of the zeroth-order Bessel function,

$$\begin{aligned} g(\mathbf{r}, t) &\sim \left(\frac{+2i\omega}{c}\right)\exp[+i\omega(t - z/c)]\frac{\exp[-i(\omega/c)(\rho^2/2z)]}{4\pi z} \\ &\quad\times \int_0^a \mathrm{d}\rho'\rho' J_0(\chi\rho')\exp[-i(\omega/c)(\rho'^2/2z)] \\ &\quad\times \int_0^{2\pi} \mathrm{d}\phi' \exp[+i(\omega/c)(\rho\rho'/z)\cos(\phi - \phi')] \\ &= \left(\frac{+2i\omega}{c}\right)\exp[+i\omega(t - z/c)]\frac{\exp[-i(\omega/c)(\rho^2/2z)]}{2z} \\ &\quad\times \int_0^a \mathrm{d}\rho'\rho' J_0(\chi\rho') J_0\left(\frac{\omega}{c}\frac{\rho}{z}\rho'\right)\exp[-i(\omega/c)(\rho'^2/2z)]. \end{aligned} \tag{3.4}$$

Letting $\rho^2 \to +iq^2 + \epsilon$, where $0 < \epsilon \ll 1$, in Eq. (6.633.2) of Ref. 26; taking the $\epsilon \to 0$ limit of that result; and using Eq. (8.406.3) of Ref. 26, one obtains

$$\int_0^\infty \exp(-iq^2x^2)J_0(\alpha x)J_0(\beta x)x\mathrm{d}x = \frac{-i}{2q^2}\exp\left(+i\frac{\alpha^2 + \beta^2}{4q^2}\right)J_0\left(\frac{\alpha\beta}{2q^2}\right). \tag{3.5}$$

Expression (3.4) then becomes

$$g(\mathbf{r}, t) \sim \exp[+i\omega(t - z/c)] \times \exp[+(i/2)(c/\omega)z\chi^2]J_0(\chi\rho) - G(\mathbf{r}, t), \tag{3.6}$$

where the remainder term

$$\begin{aligned} G(\mathbf{r}, t) &= \left(\frac{+2i\omega}{c}\right)\exp[+i\omega(t - z/c)]\frac{\exp[-i(\omega/c)(\rho^2/2z)]}{2z} \\ &\quad\times \int_a^\infty \mathrm{d}\rho'\rho' J_0(\chi\rho') J_0\left(\frac{\omega}{c}\frac{\rho}{z}\rho'\right)\exp[-i(\omega/c)(\rho'^2/2z)]. \end{aligned} \tag{3.7}$$

The second Bessel function controls the transverse variations in the integrand. If w_0 is the waist of the localized beam, then we need to restrict its argument to values within the range

$$\frac{\omega}{c}\frac{w_0 a}{z} \geq 2$$

to minimize the contributions from the remainder term G. Similarly, the exponential term in the integrand of Eq. (3.7) will also be highly oscillatory, as long as

$$z < \frac{\omega}{c}\frac{a^2}{2} = \frac{\pi a^2}{\lambda}.$$

Therefore, since localization means at least $w_0 \leq a$, we find, by combining these constraints, that in the region

$$z \leq \frac{\omega}{c}\frac{w_0 a}{2} \equiv \frac{\pi w_0 a}{\lambda} \tag{3.8}$$

the remainder term $G(\mathbf{r}, t) \approx 0$. This means that

$$g(\mathbf{r}, t) \sim \exp[+i\omega(t - z/c)]\exp[+(i/2)(c/\omega)z\chi^2]J_0(\chi\rho), \tag{3.9}$$

when the observation point is in the region defined by expression (3.8), i.e., the effective near field of the aperture. In addition, we know that

$$\frac{\omega}{c} = (\chi^2 + k_z^2)^{1/2}$$

or

$$k_z = [(\omega/c)^2 - \chi^2]^{1/2}, \tag{3.10}$$

so that, in the spectral region where $\omega/c \gg \chi$, the axial propagation constant

$$k_z \sim \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\chi^2}{(\omega/c)^2}\right] = \frac{\omega}{c} - \frac{1}{2}\frac{c}{\omega}\chi^2. \tag{3.11}$$

Since $k = \omega/c$, this is consistent with the fact that

$$k \sim k_z\left(1 + \frac{1}{2}\frac{\chi^2}{k^2}\right) \sim k_z + \frac{1}{2}\frac{\chi^2}{k_z} \tag{3.12}$$

near the axis of propagation for a localized field. Consequently, everywhere in the near-field axial region defined by expression (3.8), we obtain

$$g(\mathbf{r}, t) \sim J_0(\chi\rho)\exp[-i(k_z z - \omega t)] \equiv f(\mathbf{r}, t); \tag{3.13}$$

i.e., the original HWE solution is recovered. This result agrees with the results reported in Ref. 5 for the Bessel beam. Similarly, a backward-propagating Bessel beam produces a null contribution in this region. Thus, as with the infinite-aperture case, the forward-propagating components of a LW solution would be recovered in this near-field region. As the observation point moves into the far-field region, the remainder terms are no longer negligible and the localization effect can no longer be maintained. Specific bounds on the parameters of the resulting broad-bandwidth fields and their higher-order correlation properties are discussed in Refs. 21 and 22. For instance, the diffraction length of the field generated by an aperture that is driven with a LW solution created as a superposition of Bessel beams will be expression (3.8), with λ replaced by the effective wavelength that is associated with the set of signals driving the aperture [i.e., with the value $\lambda_{\mathrm{rad}} = 2\pi c/\omega_{\mathrm{rad}}$, where ω_{rad} is given by Eq. (2.19b) of Ref. 21].

4. RELATIONSHIP TO HILLION'S GOURSAT REPRESENTATION

Other representations of the initial-boundary-value-problem solution are possible from the characteristic-variable point of view. In particular, consider the characteristic coordinate version of the HWE (2.1):

$$\Delta_\perp f(\mathbf{r}, t) + 4\partial_\xi\partial_\eta f(\mathbf{r}, t) = 0, \tag{4.1}$$

where $\xi = z - ct$ and $\eta = z + ct$. With the ansatz that

$$f_k(\mathbf{r}, t) = \exp(ik\eta)F_k(\mathbf{r}_\perp, \xi), \tag{4.2}$$

the HWE (4.1) reduces to the Schrödinger equation[1]

$$4ik\partial_\xi F_k + \Delta_\perp F_k = 0. \tag{4.3}$$

The Green's function for this equation is derived in Appendix D and allows for positive and negative values of the characteristic variable ξ. One can then represent an axisymmetric solution of Eq. (4.1) in terms of its values on the hypersurface $\xi = 0$:

$$\begin{aligned} f_k(\rho, \xi, \eta) &= \frac{-2ik}{\xi}\exp(ik\eta)\exp(ik\rho^2/\xi) \\ &\quad\times \int_0^\infty \mathrm{d}r r\, \exp(ikr^2/\xi) J_0\left(\frac{2k}{\xi}\rho r\right) F_k(r, 0) \\ &= \frac{-2ik}{\eta}\exp(ik\xi)\exp(ik\rho^2)/\xi \\ &\quad\times \int_0^\infty \mathrm{d}r r\, \exp(ikr^2/\xi) J_0\left(\frac{2k}{\xi}\rho r\right) F_k(r, 0, 2ct). \end{aligned} \tag{4.4}$$

This recovers the form of the solution of the Goursat initial-boundary-value problem reported recently by Hillion.[23] The FWM solution[1] given by Eq. (2.12) with $z_0 = kw_0^2$ is recovered with the choice

$$F_k(r, 0) = \exp(-r^2/w_0^2). \tag{4.5}$$

Representation (4.4) is advantageous for initial data on the characteristic surfaces such as $\xi = 0$, but it has little direct applicability to physical situations that involve data on a spatial hyperplane $z = 0$. Nonetheless, it can be used to derive such a representation. In particular, for $z = 0$, Eq. (4.4) gives

$$\begin{aligned} f_k(\rho, -ct, +ct) &= \frac{2ik}{ct}\exp[ik(ct)]\exp(-ik\rho^2/ct) \\ &\quad\times \int_0^\infty \mathrm{d}r r\, \exp(-ikr^2/ct) J_0\left(\frac{2k}{ct}\rho r\right) F_k(r, 0). \end{aligned} \tag{4.6}$$

This expression is straightforwardly inverted to give

$$\begin{aligned} F_k(r, 0) &= \frac{-ik}{ct}\exp[-ik(ct)]\exp(+ikr^2/ct) \\ &\quad\times \int_0^\infty \mathrm{d}\rho\rho\, \exp(+ik\rho^2/ct) J_0\left(\frac{2k}{ct}r\rho\right) f_k(\rho, -ct, +ct). \end{aligned} \tag{4.7}$$

Therefore, inserting this result into Eq. (4.4), one obtains the following expression:

$$\begin{aligned} f_k(\rho, \xi, \eta) &= \frac{ik}{z}\exp(ik\rho^2/z)\exp(ikz)\int_0^\infty \mathrm{d}r r\, \exp(ikr^2/z) \\ &\quad\times J_0\left(\frac{2k}{z}\rho r\right) f_k(r, -ct, +ct). \end{aligned} \tag{4.8}$$

For the FWM solution,

$$f_k^{\mathrm{FWM}}(r, -ct, +ct) = \frac{z_0}{z_0 - ict}\exp(+ikct)\exp[-kr^2/(z_0 - ict)]. \tag{4.9}$$

Equation (4.8) is a rather unusual representation of a HWE solution in that it couples together the source and the observation points through the Bessel function term in the integrand. It does show that the final time history is completely governed by the initial time histories of that solution on the plane $z = 0$. Moreover, the propagation terms in this exact result are identical to those encountered in a paraxial or a Fresnel approximation. However, in contrast to Hillion's conclusions,[23] Eq. (4.8), which is equivalent to Eq. (4.4), does not answer any issues concerning the causality of the LW solutions of Eq. (2.1); i.e., it does not separate the causal and the anticausal contributions as we have done in Section 2. It simply generates the entire solution from its boundary values on the hypersurface $z = 0$.

5. SLINGSHOT SUPERLUMINAL PULSES

The bidirectional representation introduced in Ref. 3 makes use of a factorization of the wave operator in terms of the characteristic light-cone variables $\zeta = z - ct$ and $\eta = z + ct$. Nonetheless, the HWE and this factorization do not discriminate against pulses whose group speed $v_g < c$ or $v_g > c$. These subluminal or superluminal pulses represent interference patterns that have been constructed from basic building blocks each traveling at the speed of light c. We discuss the superluminal case below; the subluminal case follows in an analogous fashion.

Let us consider superluminal solutions of the wave and Maxwell's equations. We first make an analogy with the original description of FWM's as projections onto real space of field solutions that have moving complex source locations.[1] The field created by an electron moving at a constant speed v along the z axis in a medium of index of refraction n is readily obtained from well-known formulas in free space.[27,28] The scalar and vector Liénard–Wiechert potentials (in cgs units),

$$\phi(\mathbf{r}, t) = \frac{q}{\{[1 - (v/c_m)^2](x^2 + y^2) + (z - vt)^2\}^{1/2}}, \tag{5.1}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c_m} \phi(\mathbf{r}, t), \tag{5.2}$$

where the speed of light in the medium $c_m = c/n$ and the velocity along the z axis $\mathbf{v} = v\hat{z}$, result in the electromagnetic fields

$$\mathbf{E}(\mathbf{r}, t) = q \frac{[1 - (v/c_m)^2][x\hat{x} + y\hat{y} + (z - vt)\hat{z}]}{\{[1 - v/c_m)^2](x^2 + y^2) + (z - vt)^2\}^{3/2}}, \tag{5.3}$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{\mathbf{v}}{c_m} \times \mathbf{E}(\mathbf{r}, t). \tag{5.4}$$

Now let the electron move along the complex z axis displaced from real space by the distance iz_0 so that the source singularity, rather than being at $(x, y, z) = (0, 0, vt)$, is at $(x, y, z) = (0, 0, vt + iz_0)$. The resulting potentials and fields are

$$\phi(\mathbf{r}, t) = \frac{q}{\{[1 - (v/c_m)^2](x^2 + y^2) + [(z - vt) - iz_0]^2\}^{1/2}}, \tag{5.5}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c_m} \phi(r, t), \tag{5.6}$$

$$\mathbf{E}(\mathbf{r}, t) = q \frac{[1 - (v/c_m)^2]\{x\hat{x} + y\hat{y} + [(z - vt) - iz_0]\hat{z}\}}{\{[1 - (v/c_m)^2](x^2 + y^2) + [(z - vt) - iz_0]^2\}^{3/2}}, \tag{5.7}$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{\mathbf{v}}{c_m} \times \mathbf{E}(\mathbf{r}, t). \tag{5.8}$$

The projections of these solutions onto real space–time are purely translational, maintaining their form over all space–time about the group velocity center $z = vt$. If these pulses are subluminal, i.e., if $v/c_m < 1$, these potentials and fields have singularities in the plane $z = vt$ in real space along the ring $\rho = (x^2 + y^2)^{1/2} = z_0[1 - (v/c_m)^2]^{-1/2}$. On the other hand, if the pulses are superluminal, i.e., if $v/c_m > 1$, these potentials and fields have no singularities in real space, the ring singularity being displaced to a complex location $\rho = iz_0[(v/c_m)^2 - 1]^{-1/2}$ and $z = vt$. Thus one obtains interesting free-space, nonsingular HWE solutions (in real space–time) but at a cost of their being superluminal. The energy associated with this superluminal class of solutions is infinite. Consider the HWE solution $f(\rho, \theta, z - vt)$:

$$\begin{aligned}[\nabla^2 - (1/c_m)^2 \partial_t^2] f(\rho, \theta, z - vt) &= \partial_\rho^2 f + \frac{1}{\rho}\partial_\rho f + \frac{1}{\rho^2}\partial_\theta^2 f + \partial_z^2 f - \left(\frac{1}{c_m}\right)^2 \partial_t^2 f \\ &= \partial_\rho^2 f + \frac{1}{\rho}\partial_\rho f + \frac{1}{\rho^2}\partial_\theta^2 f + \gamma^{-2}\partial_\zeta^2 f = 0,\end{aligned} \tag{5.9}$$

where

$$\zeta = z - vt \tag{5.10}$$

$$\gamma^{-2} = 1 - (v/c_m)^2. \tag{5.11}$$

If one sets $\tau = \gamma\zeta$ and $v/c_m < 1$, then Eq. (5.9) becomes a three-dimensional Laplace equation:

$$\tilde{\Delta} f = 0, \tag{5.12}$$

where the modified Laplacian $\tilde{\Delta}$ is with respect to the variables (ρ, θ, τ). The basis solutions to this equation cannot have compact support, existing over all space, and hence will not have finite energy. Similarly, taking $v/c_m > 1$ and $\tau = -i|\gamma|\zeta$, we reduce Eq. (5.9) to the two-dimensional wave equation

$$\Delta_\perp f - \partial_\tau^2 f = 0. \tag{5.12'}$$

The resulting two-dimensional basis waves also exhibit infinite energy. If $v = c$, then the time-derivative term disappears, and one has $\Delta_\perp f = 0$, which requires f to be harmonic in the transverse coordinates so that it must have noncompact support. Thus one can achieve finite energy superluminal pulses only by taking superpositions of any of these basis solutions over their free parameters.

One can write the wave equation in yet another form that specifically isolates the $z - vt$ dependence. Let $\eta = \rho/\gamma$. Then Eq. (5.9) can be written explicitly in the form

$$\partial_\eta^2 \Phi + \frac{1}{\eta}\partial_\eta \Phi + \frac{1}{\eta^2}\partial_\theta^2 \Phi + \partial_\zeta^2 \Phi = 0. \tag{5.13}$$

Consider as a general form of the solutions to this equation,

$$\Phi_{nm}(\eta, \theta, \zeta) = \frac{h_{nm}(\eta, \zeta)}{(\eta^2 + \xi^2)^{m/2}} \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}, \tag{5.14}$$

where $\xi = \zeta + iz_0$, $m = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. Simple insertion of Eq. (5.14) into Eq. (5.13) and collection of like terms lead to the equations that must be satisfied by h_{nm}:

$$\eta^2(\partial_\eta{}^2 h_{nm} + \partial_\zeta{}^2 h_{nm}) = n^2 h_{nm} - \eta\partial_\eta h_{nm}, \quad (5.15a)$$

$$\eta\partial_\eta h_{nm} + \zeta\partial_\zeta h_{nm} = \left(\frac{m-1}{2}\right) h_{nm}. \quad (5.15b)$$

The simplest case is the azimuthally symmetric one, where $n = 0$, $m = 1$, and h_{00} a complex constant:

$$\Phi_{00}(\eta, \theta, \zeta) = \frac{h_{00}}{(\eta^2 + \xi^2)^{1/2}}. \quad (5.16)$$

Consider the superluminal case and set $h_{00} = iz_0$ and $\gamma^{-1} = -i\kappa$, where the real constant $\kappa \geq 1$. One then has

$$\Phi_{00}(\eta, \theta, \zeta) = \frac{iz_0}{\{-(\kappa\rho)^2 + [(z - vt) + iz_0]^2\}^{1/2}} = \frac{z_0}{\{(\kappa\rho)^2 + [z_0 - i(z - vt)]^2\}^{1/2}}. \quad (5.17)$$

This is equivalent, modulo a constant, to the electron potential solution [Eq. (5.5)]. Note that like Eq. (5.5) this solution is strictly unidirectional. The energy of this solution diverges logarithmically:

$$\int_{-\infty}^{\infty} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}\rho\rho|\Phi_{00}|^2 \propto \int_0^{\infty} \frac{\mathrm{d}\rho\rho}{\{[\gamma^{-2}\rho^2 + (z - vt)^2 - z_0{}^2]^2 + 4z_0{}^2(z - vt)^2\}^{1/2}}$$
$$= \frac{\gamma^2}{2} \ln\{\xi + [\xi^2 + 4z_0{}^2(z - vt)^2]^{1/2}\}\Big|_{\xi=(z-vt)^2-z_0{}^2}^{\xi=\infty} = \infty. \quad (5.18)$$

The intensity pattern of HWE solution (5.17) with speed $v \simeq c$ in a medium with index of refraction $n > 1$ has the characteristic rabbit-head-shaped pattern of the LW solutions (the center of the pulse being the head, the tail regions the ears). Despite the localization of the intensity, the result [relation (5.18)] indicates that the energy in the ear-shaped regions of this solution diverges logarithmically.

Higher-order solutions have energies that diverge algebraically. It is realistic to expect finite energy superpositions of these basis solutions. Transverse electric fields can also be defined simply from the solution Φ_{00} by introduction of the alternative vector potential $\mathbf{A} = \Phi_{00}\hat{\theta}$ and the scalar potential $\phi = 0$. The electric field is then $\mathbf{E} = -\partial_{ct}\mathbf{A}$. It is a purely transverse-to-z electric field that now has its maximum in its imaginary component rather than in its real component in the plane $z = vt$ when $\rho = 0$. Note that it is different from solution (5.7), which has its longitudinal component nonzero in any plane other than $z = vt$.

In vacuum, the solutions given by Eq. (5.17) strictly have speeds $v > c$. However, in a medium, the speed can be adjusted to satisfy the standard restrictions $c > v > c_m$. Superluminal fields, e.g., Čherenkov radiation, are known to exist in a medium. A major difference between the LW solutions (5.17), for instance, and Čherenkov radiation in this velocity regime is the fact that the central peak dominates the field intensity and that there are forward and backward tails rather than just a forward Mach cone. The forward tails are actually in the direction of the Mach cone. Approximate forms of these superluminal waves could in principle be found in a medium. These solutions might be of additional interest to tachyon research and to other potential faster-than-light effects, such as the observed superluminal emission jets from extragalactic radio sources.[29]

We have made several comparisons of these scalar and electromagnetic superluminal solutions with the other LW's, such as the FWM's. The scalar versions had similar characteristics in that they had intense central peaks and long-lasting tail regions as well as frequency spectra that had a high-pass character to them. The time histories associated with Eq. (5.17) are in general less complex than those associated with the FWM's, e.g., $Re\Phi_{00}(\rho = 0, \theta, z = 0, t) = 1/[1 + (vt/z_0)^2]^{1/2}$. With the results of Section 3, the beam generated by driving a finite aperture with the superluminal solutions [Eq. (5.17)] can be characterized.

Driving a circular array of radius a with one of these superluminal HWE solutions that has a waist w when $z = t = 0$ and a maximum frequency of interest f_{rad}, one generates a beam that recovers this solution only in its near field, i.e., for distances given by expression (3.8): $z < \pi a w f_{\mathrm{rad}}/c \sim \pi a w/\lambda_{\mathrm{rad}}$. The superluminal pulse beam looks like a slingshot effect. The tails of the superluminal solution first drive the array, establishing a background field moving at speed c. A moving interference pattern representing the central peak then moves out from the center at speed $v > c$ and catches up to the background field waves. The superluminal interference center then disappears as this pulse center outruns the background waves, i.e., the interference pattern disappears as it outruns its component waves. (See the Note Added in Proof).

Many other properties of the superluminal pulse are in fact analogous to other LW solutions. This is to be expected, since these HWE solutions are obtainable as superpositions of Bessel beams from their corresponding bidirectional representation, even though they are unidirectional. The bidirectional representation of Eq. (5.17) is straightforward. In particular, set $\zeta = z - c_m t$ and $\eta = z + c_m t$. The bidirectional spectrum,

$$C(\chi, \alpha, \beta) = \frac{2i}{\pi} \frac{2z_0}{1 + v/c_m} \frac{\exp\{-[2z_0/(1 + v/c_m)]\alpha\}}{\chi^2 - 4\alpha\beta} \times \delta\left[\beta + \alpha\left(\frac{1 - v/c_m}{1 + v/c_m}\right)\right], \quad (5.19)$$

and the bidirectional representation,

$$f(\rho, \theta, z, t) = \int_0^{\infty} d\chi\chi \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \mathrm{d}\beta C(\chi, \alpha, \beta) J_0(\chi\rho) \times \exp(-i\alpha\zeta)\exp(i\beta\eta)\delta(\alpha\beta - \chi^2/4), \quad (5.20)$$

yield Eq. (5.17), i.e.,

$$\begin{aligned} f(\rho,\theta,z,t) &= \frac{2i}{\pi}\frac{2z_0}{1+v/c_m}\int_0^\infty \mathrm{d}\chi\chi\int_{-\infty}^\infty \mathrm{d}\alpha \frac{\exp(-i\alpha\zeta)\exp\{-i\alpha[(1-v/c_m)/(1+v/c_m)]\eta\}\exp\{-\alpha[2z_0/(1+v/c_m)]\}}{\chi^2+4\alpha^2[(1-v/c_m)/(1+v/c_m)]}J_0(\chi\rho) \\ &= \frac{4i}{\pi}\frac{2z_0}{1+v/c_m}\int_0^\infty \mathrm{d}\alpha \cos\left\{\alpha\left[\zeta+\eta\left(\frac{1-v/c_m}{1+v/c_m}\right)-i\frac{2z_0}{1+v/c_m}\right]\right\}\int_0^\infty \mathrm{d}\chi\chi\frac{J_0(\chi\rho)}{\chi^2+4\alpha^2[(1-v/c_m)/(1+v/c_m)]} \\ &= \frac{4i}{\pi}\frac{2z_0}{1+v/c_m}\int_0^\infty \mathrm{d}\alpha K_0\left[2\alpha\left(\frac{1-v/c_m}{1+v/c_m}\right)^{1/2}\rho\right]\cos\left\{\alpha\left[\zeta+\eta\left(\frac{1-v/c_m}{1+v/c_m}\right)-i\frac{2z_0}{1+v/c_m}\right]\right\} \\ &= i\frac{2z_0}{1+v/c_m}\left\{4\left(\frac{1-v/c_m}{1+v/c_m}\right)\rho^2+\left[\zeta+\eta\left(\frac{1-v/c_m}{1+v/c_m}\right)-i\frac{2z_0}{1+v/c_m}\right]^2\right\}^{-1/2} \\ &= \frac{iz_0}{\{\gamma^{-2}\rho^2+[\zeta(1+v/c_m)/2+\eta(1-v/c_m)/2-iz_0]^2\}^{1/2}} \\ &= \frac{iz_0}{[\gamma^{-2}\rho^2+(z-vt-iz_0)^2]^{1/2}} \equiv \Phi_{00}(\eta,\theta,\zeta), \end{aligned} \tag{5.21}$$

where the last equivalence has been obtained as above with the assumption that $v > c_m$. Similarly, electron potential (5.1), an inhomogeneous wave-equation solution that is strictly forward propagating, is obtained straightforwardly from the bidirectional spectrum

$$C(\chi,\alpha,\beta) = \frac{4}{\pi}\frac{1}{1+v/c_m}\frac{q}{\chi^2-4\alpha\beta}\delta\left[\beta+\alpha\left(\frac{1-v/c_m}{1+v/c_m}\right)\right]. \tag{5.22}$$

Thus, even though it constructs solutions from products of forward- and backward-traveling plane waves propagating at speed c_m, the bidirectional representation can produce unidirectional subluminal and superluminal solutions of the homogeneous and the inhomogeneous scalar-wave equations.

The high-pass filter property of the spectra associated with the HWE pulse solution (5.17) is also a common characteristic. As the waist of the modified-power-spectrum (MPS) pulse[2] is directly related to its lowest and highest frequencies, one finds that the waist of solution (5.17), is related to its highest radian frequency of relevance $\omega_{\max} = c/z_0$. This relation appears to be due to the exponential increase of the spectrum at the lower frequencies in the MPS pulse and to an exponential rolloff in both the MPS and slingshot pulses at higher frequencies. The complete solution is obtained only with an infinite (extremely large in the MPS pulse case, infinite for the FWM) aperture, and it is recovered approximately by a finite aperture in its near field, as discussed above in Section 3. For a finite array, the resulting slingshot pulse beam actually has the same form as the beam that was obtained in the original MPS pulse-driven array experiments:[14] a rabbit-head shape in the near field; a central peak with only lagging tails on the edge of the near-to-far-field boundary; and a spherically expanding beam in the far field. The tails of the beam that is generated by the pulse-driven array replace the sidelobes associated with conventional cw-driven aperture patterns. The decrease in the size of the sidelobes by an increase in the bandwidth is common to beams that are generated by arrays driven with LW solutions.

6. CONCLUSIONS

We have shown that if an aperture is driven with an arbitrary HWE solution, only the forward-propagating components of that solution can be reconstructed from its values on an open initial-boundary-value surface. An infinite aperture affords one the ability to re-create those components everywhere in the positive half-space; a finite aperture allows one to re-create the components everywhere in the near-field region. Since the initial driving signals in an aperture correspond simply to a set of time histories, the presence of both the positive and the negative values of k_z and frequencies ω are permitted. The propagator in the Huygens representation, being strictly causal, filters out the acausal components in those regions. Thus, by designing LW solutions with minimal backward propagating components, as was accomplished for the acoustic experiments reported in Refs. 14 and 15, and using them to drive an aperture, one can recreate fields that are remarkably close approximations of those solutions. One can then obtain for application purposes some of the highly desirable localization properties of beams that are generated by driving an array with an ultrawide bandwidth set of pulses, as described in Refs. 21 and 22. Bevensee[30] has recently generalized the results we have reported here to LW solutions of Maxwell's equations.

The HWE solution reconstruction results were applied specifically to a class of superluminal beams. The slingshot field that was generated by driving an aperture with one of these solutions is characterized as a moving interference pattern. The intensity of this field has a peak that moves at speeds $v_g > c$, even though its constituent signals are traveling at the characteristic wave speed of the medium. The effect is present in the near field of the aperture; it is lost once the interference pattern reaches the far field.

APPENDIX A: BEHAVIOR OF SOLUTIONS RECONSTRUCTED BY THE HUYGENS REPRESENTATION ON THE HEMISPHERE AT INFINITY

Implicit in the Huygens representation is the behavior of the HWE solution under reconstruction on the sphere at infinity. In the frequency domain this behavior is governed by Sommerfeld's radiation condition. The generalization of that condition to pulse solutions is considered here.

Consider Green's theorem:

$$g(\mathbf{r}, t) = \int_{\Sigma} \mathrm{d}S' \frac{\Psi(x', y', z' = 0, t - R/c)}{4\pi R} + \int_{R_\infty} \mathrm{d}S' \frac{\overline{\Psi}(x', y', z', t - R/c)}{4\pi R}, \qquad \text{(A1)}$$

where Σ represents the plane $z = 0$, R_∞ represents the hemisphere at infinity, Ψ is given by Eq. (2.2b), and

$$\overline{\Psi}(x', y', z', t - R/c) = \hat{n}_\infty \cdot \left\{[\nabla' f] - [\partial_{ct'} f]\frac{\hat{R}}{R} - [f]\frac{\hat{R}}{R^2}\right\}, \qquad \text{(A2)}$$

where the unit vector $\hat{R} = (\mathbf{r} - \mathbf{r}')/R$ and $\hat{n}_\infty$ is the outwardly pointing unit normal to the surface R_∞.

We obtain the Huygens representation [Eqs. (2.2a)] of a HWE solution in the region $z > 0$ from Eq. (A1), assuming that any contributions from the surface R_∞ are negligible, i.e., that the second integral is zero. In Eq. (A1) we also assume that, everywhere in the region bounded by Σ and R_∞, f and $\partial_t f$ are zero at $t = 0$ and f and $\partial_t f$ are continuous. Mathematically, the Green's representation does not impose any causality condition on the behavior of the HWE solution but allows solutions to come into the region from either surface R_∞ or Σ. Causality is imposed as an additional constraint on the overall behavior of the solution to make it physically realizable. In the time domain this condition is expressed as a restriction only to outward-propagating waves on R_∞. The mathematical representation of this condition is derived as follows:

If $\mathrm{d}\Omega$ is the solid angle subtended by R_∞, then the integral

$$\int_{R_\infty} \mathrm{d}S' \frac{\overline{\Psi}(x', y', z', t - R/c)}{4\pi R} = \lim_{R\to\infty} \int \mathrm{d}\Omega R^2 \frac{\overline{\Psi}(x', y', z', t - R/c)}{4\pi R}. \qquad \text{(A3)}$$

If we assume that the HWE solution f is regular at infinity, i.e., that $\lim_{R\to\infty}(Rf) \sim$ constant, the integral

$$\lim_{R\to\infty} \int (\hat{n}_\infty \cdot \hat{R})[f]\mathrm{d}\Omega \to 0. \qquad \text{(A4)}$$

Furthermore, since $|\mathbf{r}'| > |\mathbf{r}|$ on R_∞, the unit vector $\hat{R}$ has a negative orientation with respect to $\hat{n}_\infty$, i.e., $\hat{n}_\infty \cdot \hat{R} < 0$. The integral (A3) is then zero only if the HWE solution f represents an outward-propagating wave on the surface R_∞, i.e., only if it satisfies the relation

$$\lim_{R\to\infty}\{R[(\hat{n}_\infty \cdot \nabla')f - (\hat{n}_\infty \cdot \hat{R})\partial_{ct'} f]\} \to 0. \qquad \text{(A5)}$$

Thus, if the HWE solution f is forward propagating, the Huygens representation (2.2) will reconstruct it in the region $z > 0$, i.e., $g \equiv f$.

APPENDIX B: BESSEL BEAMS AND THE HUYGENS REPRESENTATION

The action of the Huygens representation (2.2) on the Bessel beams

$$f(\rho, z, t) = J_0(\chi\rho)\exp[\pm i(k_z z \mp \omega t)], \qquad \text{(B1)}$$

where $k_z = [(\omega/c)^2 - \chi^2]^{1/2}$ and $k = \omega/c$, is considered. In particular, let f_+ be the forward-propagating beam

$$f_+(\rho, z, t) = J_0(\chi\rho)\exp[+i(k_z z - \omega t)],$$

and let the Huygens operator $\mathscr{H}_\Sigma$ be taken from Σ, the infinite plane $z = 0$. One then has

$$\mathscr{H}_\Sigma[f_+] = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^\infty \mathrm{d}\rho'\rho'\frac{1}{R} \times \left\{-[\partial_{z'} f_+] + [\partial_{ct'} f_+]\frac{z}{R} + [f_+]\frac{z}{R^2}\right\}$$
$$= \frac{\exp(-i\omega t)}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_0^\infty \mathrm{d}\rho'\rho'\left\{-ik_z - i\frac{\omega}{c}\frac{z}{R} + \frac{z}{R^2}\right\} \times \frac{\exp[i(\omega/c)R]}{R}J_0(\chi\rho'). \qquad \text{(B2)}$$

Since

$$\partial_z\left\{\frac{\exp[i(\omega/c)R]}{R}\right\} = \left(\frac{i\omega}{c}\frac{z}{R^2} - \frac{z}{R^3}\right)\exp[i(\omega/c)R], \qquad \text{(B3)}$$

relation (B2) can be expressed as

$$\mathscr{H}_\Sigma[f_+] = \frac{\exp(-i\omega t)}{4\pi}(-ik_z\mathscr{X}_+ - \partial_z\mathscr{X}_+), \qquad \text{(B4)}$$

where the integral term

$$\mathscr{X}_\pm = \int_0^{2\pi}\mathrm{d}\phi'\int_0^\infty \mathrm{d}\rho'\rho'\frac{\exp[\pm i(\omega/c)R]}{R}J_0(\chi\rho'). \qquad \text{(B5)}$$

With

$$R^2 = \rho'^2 + \rho^2 - 2\rho\rho'\cos\phi' + z^2, \qquad \text{(B6)}$$

one has the identity

$$\frac{\exp(\pm ikR)}{R} = \pm i\int_0^\infty \frac{\lambda\mathrm{d}\lambda}{(k^2 - \lambda^2)^{1/2}}\exp[\pm i(k^2 - \lambda^2)^{1/2}|z|] \times J_0[\rho'^2 + \rho^2 - 2\rho\rho'\cos\phi')^{1/2}], \qquad \text{(B7)}$$

which follows directly from the Sommerfeld identity given in Ref. 31, p. 435. Inserting Eq. (B7) directly into Eq. (B5) gives

$$\mathscr{X}_\pm = \pm i\int_0^\infty \frac{\lambda\mathrm{d}\lambda}{[(\omega/c)^2 - \lambda^2]^{1/2}}\exp\{\pm i[(\omega/c)^2 - \lambda^2]^{1/2}|z|\} \times \int_0^\infty \mathrm{d}\rho'\rho' J_0(\chi\rho') \times \int_0^{2\pi}\mathrm{d}\phi' J_0[\lambda(\rho'^2 + \rho^2 - 2\rho\rho'\cos\phi')^{1/2}]. \qquad \text{(B8)}$$

With the identity given by Eq. (8.530.2) of Ref. 26,

$$J_0[\lambda(\rho'^2 + \rho^2 - 2\rho\rho'\cos\phi')^{1/2}] = \sum_{l=-\infty}^{\infty} J_l(\lambda\rho)J_l(\lambda\rho')\exp(il\phi'), \qquad \text{(B9)}$$

the ϕ' integration of Eq. (B8) yields

$$\int_0^{2\pi}\mathrm{d}\phi' J_0[\lambda(\rho'^2 + \rho^2 - 2\rho\rho'\cos\phi')^{1/2}] = \sum_{l=-\infty}^{\infty} J_l(\lambda\rho)J_l(\lambda\rho')\int_0^{2\pi}\mathrm{d}\phi'\exp(il\phi') = 2\pi J_0(\lambda\rho)J_0(\lambda\rho'), \qquad \text{(B10)}$$

and Eq. (B8) becomes

$$\mathscr{Z}_{\pm} = \pm 2\pi i \int_0^{\infty} \frac{\lambda \mathrm{d}\lambda}{[(\omega/c)^2 - \lambda^2]^{1/2}} \exp\{\pm i[(\omega/c)^2 - \lambda^2]^{1/2}|z|\} \times J_0(\chi\rho) \int_0^{\infty} \mathrm{d}\rho'\rho' J_0(\chi\rho') J_0(\lambda\rho'). \quad \text{(B11)}$$

Since

$$\int_0^{\infty} \mathrm{d}\rho'\rho' J_0(\lambda\rho') J_0(\chi\rho') = \frac{\delta(\lambda - \chi)}{\lambda}, \quad \text{(B12)}$$

Eq. (B11) reduces immediately to the result that

$$\mathscr{Z}_{\pm} = \pm 2\pi i \frac{\exp(\pm i k_z |z|)}{k_z} J_0(\chi\rho). \quad \text{(B13)}$$

Alternately, as shown by Donnelly[32] and suggested by one of the reviewers, one can rewrite Eq. (B5) as a two-dimensional convolution:

$$\mathscr{Z}_{\pm}(\rho, z) = J_0(\chi\rho) * \frac{\exp[\pm i(\omega/c)(\rho^2 + z^2)^{1/2}]}{(\rho^2 + z^2)^{1/2}}. \quad \text{(B14)}$$

A two-dimensional Fourier transform of this result gives

$$\mathscr{F}_{x,y}\{\mathscr{Z}_{\pm}\}(k_x, k_y) = \mathscr{F}_{x,y}\{J_0(\chi\rho)\}(\kappa) \times \mathscr{F}_{x,y}\left\{\frac{\exp[\pm i(\omega/c)(\rho^2 + z^2)^{1/2}]}{(\rho^2 + z^2)^{1/2}}\right\}(\kappa), \quad \text{(B15)}$$

where $\kappa^2 = k_x{}^2 + k_y{}^2$. Since

$$\mathscr{F}_{x,y}\{J_0(\chi\rho)\}(\kappa) = 2\pi \frac{\delta(\kappa - \chi)}{\kappa}, \quad \text{(B16)}$$

inverse Fourier transforming Eq. (B15) gives

$$\mathscr{Z}_{\pm}(\rho, z) = J_0(\chi\rho) \times \mathscr{F}_{x,y}\left\{\frac{\exp[\pm i(\omega/c)(\rho^2 + z^2)^{1/2}]}{(\rho^2 + z^2)^{1/2}}\right\}(\chi). \quad \text{(B17)}$$

The remaining two-dimensional Fourier transform

$$\mathscr{F}_{x,y}\left\{\frac{\exp[\pm i(\omega/c)(\rho^2 + z^2)^{1/2}]}{(\rho^2 + z^2)^{1/2}}\right\} = \pm 2\pi i \frac{\exp\{\pm i|z|[(\omega/c)^2 - \chi^2]^{1/2}\}}{[(\omega/c)^2 - \chi^2]^{1/2}} \equiv \pm 2\pi i \frac{\exp(\pm i k_z |z|)}{k_z}, \quad \text{(B18)}$$

so that with Eq. (B17) the final relation [Eq. (B13)] is again recovered. Equation (B13) also means that

$$\partial_z \mathscr{Z}_{\pm} = -2\pi \operatorname{sgn}(z) \exp(\pm i k_z |z|)^{1/2} J_0(\chi\rho). \quad \text{(B19)}$$

For observation points in the region $z > 0$, Eq. (B4) then yields

$$\mathscr{H}_{\Sigma}[f_+] = \frac{\exp(-i\omega t)}{4\pi}\left[-ik_z \frac{2\pi i}{k_z} - (-2\pi)\right] \exp(ik_z z) J_0(\chi\rho) = \exp[i(k_z z - \omega t)] J_0(\chi\rho) \equiv f_+. \quad \text{(B20)}$$

Similarly, if

$$f_-(\rho, z, t) = J_0(\chi\rho) \exp[+i(k_z z + \omega t)], \quad \text{(B21)}$$

one then has

$$\begin{aligned} \mathscr{H}_{\Sigma}[f_-] &= \frac{\exp(+i\omega t)}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\infty} \mathrm{d}\rho'\rho' \times \left\{-ik_z + i\frac{\omega}{c}\frac{z}{R} + \frac{z}{R^2}\right\} \frac{\exp[-i(\omega/c)R]}{R} J_0(\chi\rho'), \\ &= \frac{\exp(+i\omega t)}{4\pi}[-ik_z \mathscr{Z}_- - \partial_z \mathscr{Z}_-], \\ &= \frac{\exp(+i\omega t)}{4\pi}\left[-ik_z \frac{-2\pi i}{k_z} - (-2\pi)\right] \exp(ik_z z) J_0(\chi\rho), \\ &= \frac{1}{4\pi}[-2\pi - (-2\pi)] \exp[i(k_z z + \omega t)] J_0(\chi\rho) \equiv 0. \end{aligned} \quad \text{(B22)}$$

Similarly, if

$$f_{\pm}(\rho, z, t) = J_0(\chi\rho) \exp[-i(k_z z \mp \omega t)], \quad \text{(B23)}$$

one has

$$\mathscr{H}_{\Sigma}[f_+] \equiv f_+ \quad \text{(B24)}$$

$$\mathscr{H}_{\Sigma}[f_-] \equiv 0. \quad \text{(B25)}$$

It is then readily shown (in the limit that $\chi \to 0$) that

$$\mathscr{H}_{\Sigma}\{\exp[\pm i(k_z z - \omega t)]\} \equiv \exp[\pm i(k_z z - \omega t)] \quad \text{(B26)}$$

$$\mathscr{H}_{\Sigma}\{\exp[\pm i(k_z z + \omega t)]\} \equiv 0. \quad \text{(B27)}$$

Thus the Huygens operator $\mathscr{H}_{\Sigma}$ passes only forward-propagating beams or plane waves, and hence any linear combination of them, into the region $z > 0$. On the other hand, Eqs. (B13) and (B19) also show that analogous results hold in the region $z < 0$:

$$\mathscr{H}_{\Sigma}\{J_0(\chi\rho)\exp[\mp i(k_z z + \omega t)]\} \equiv -J_0(\chi\rho)\exp[\mp i(k_z z + \omega t)] \quad \text{(B28)}$$

$$\mathscr{H}_{\Sigma}\{J_0(\chi\rho)\exp[\mp i(k_z z - \omega t)]\} \equiv 0. \quad \text{(B29)}$$

Combining Eqs. (B20)–(B22) or (B23)–(B25) with Eqs. (B28) and (B29), one obtains the extinction theorem (see, for instance, Ref. 31, pp. 500–501, or Ref. 33, pp. 101–102) for these Bessel beams; i.e., the field scattered from Σ cancels the incident field in the source region $z < 0$ and produces the incident field in the observation region $z > 0$.

APPENDIX C: FOURIER TRANSFORM OF THE FOCUS-WAVE-MODE PULSE

The desired Fourier transform of the FWM pulse [Eq. (2.12)] is defined by expression (2.13):

$$\hat{f}_{\mathrm{FWM}}(\rho, z, \omega) = \int_{-\infty}^{\infty} \mathrm{d}t \exp(-i\omega t) f_{\mathrm{FWM}}(\rho, z - ct, z + ct). \quad \text{(C1)}$$

With the series definition of the exponential function, one can rewrite Eq. (C1) as

$$\hat{f}_{\mathrm{FWM}}(\rho, z, \omega) = z_0 \exp(ikz) \sum_{n=0}^{\infty} \frac{(-k\rho^2)^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t \times \exp[-i(\omega - kc)t] \frac{1}{[(z_0 + iz) - ict)]^{n+1}}. \quad \text{(C2)}$$

However, with Eq. (3.382.7) of Ref. 26:

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{\exp(-ipx)}{(\xi - ix)^{n+1}} = H(p)\frac{2\pi p^n \exp(-\xi p)}{n!}, \tag{C3}$$

where $H(p)$ is Heaviside's function, and with the series definition of the Bessel function [Eq. (8.402) of Ref. 26],

$$J_0(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!\,n!}, \tag{C4}$$

Eq. (C2) becomes

$$\begin{aligned}\hat{f}_{\mathrm{FWM}}(\rho, z, \omega) &= \frac{2\pi z_0}{c}\exp(ikz)H(\omega - kc)\\ &\quad\times \exp\{-[(z_0 + iz)/c](\omega - kc)\}\\ &\quad\times \sum_{n=0}^{\infty} \frac{(-k\rho^2)^n}{n!}\frac{(\omega - kc)^n}{c^n n!}\\ &= \frac{2\pi z_0}{c}H(\omega - kc)\exp[k(z_0 + 2iz)]\\ &\quad\times \exp[-\omega(z_0 + iz)/c]\\ &\quad\times J_0\left\{2\rho\left[k\left(\frac{\omega}{c} - k\right)\right]^{1/2}\right\}.\end{aligned} \tag{C5}$$

In the same manner, one has

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}t \exp(-i\omega t) f_{\mathrm{FWM}}^*(\rho, z - ct, z + ct)\\ &= \frac{2\pi z_0}{c}H[-(\omega + kc)]\exp[k(z_0 + 2iz)]\\ &\quad\times \exp[+\omega(z_0 + iz)/c]J_0\left\{2\rho\left[-k\left(\frac{\omega}{c} + k\right)\right]^{1/2}\right\}\\ &\equiv \hat{f}_{\mathrm{FWM}}(\rho, z, -\omega).\end{aligned} \tag{C6}$$

APPENDIX D: GREEN'S FUNCTION FOR SCHRÖDINGER EQUATION IN R^{2+1}

Consider the Schrödinger equation

$$4ik\partial_\xi F_k + \Delta_\perp F_k = 0, \tag{D1}$$

where the variable ξ is allowed to take positive and negative values. The following Green's function derivation is attributable to Donnelly.[32]

A three-dimensional Fourier transform representation of an axisymmetric solution of Eq. (D1) is

$$\begin{aligned}F_k(\rho, \xi) &= \int_{-\infty}^{\infty} \mathrm{d}k_\xi \exp(i2\pi k_\xi \xi)\\ &\quad\times \int_{R^2} \mathrm{d}\boldsymbol{\kappa} \exp(i2\pi\boldsymbol{\kappa}\cdot\mathbf{r}_\perp)\Lambda(|\boldsymbol{\kappa}|, k_\xi)\delta(2k_\xi + \pi|\boldsymbol{\kappa}|^2/k)\\ &= 2\pi\int_0^{\infty} \mathrm{d}\kappa\kappa J_0(2\pi\kappa\rho)\Lambda(\kappa, \pi\kappa^2/k)\exp(-i\pi^2\xi\kappa^2/k).\end{aligned} \tag{D2}$$

We obtain the corresponding Green's function $G_k(\rho, \xi)$ by setting $\Lambda(\boldsymbol{\kappa}, k_\xi) = 1$. Then, with Eq. (6.631.6) of Ref. 26, one has for $\xi > 0$

$$G_k(\rho, \xi) = \frac{-ik}{\pi}\frac{\exp(ik\rho^2/\xi)}{\xi} \tag{D3a}$$

and for $\xi < 0$

$$G_k(\rho, \xi) = \frac{ik}{\pi}\frac{\exp(-ik\rho^2/|\xi|)}{|\xi|}, \tag{D3b}$$

so that for $\xi \neq 0$

$$G_k(\rho, \xi) = \frac{-ik}{\pi}\frac{\exp(ik\rho^2/\xi)}{\xi}. \tag{D3c}$$

Finally, for $\xi = 0$ one has explicitly

$$G_k(\rho, \xi = 0) = \frac{\delta(\rho)}{2\pi\rho}. \tag{D4}$$

ACKNOWLEDGMENTS

The authors thank two unnamed reviewers for their detailed comments and suggestions, which led to this improved manuscript. This research was performed in part by the Lawrence Livermore National Laboratory under the auspices of the U.S. Department of Energy under contract W-7405-ENG-48.

Note Added in Proof: Since submission of this manuscript, the authors have found that J.-Y. Lu and J. F. Greenleaf of the Mayo Clinic Foundation experimentally verified this slingshot pulse behavior with ultrasound waves in water: J.-Y. Lu and J. F. Greenleaf, "Experimental verification of nondiffracting X waves," IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 441–446 (1992).

REFERENCES

1. R. W. Ziolkowski, "Exact solutions of the wave equation with complex source locations," J. Math. Phys. **26**, 861–863 (1985).
2. R. W. Ziolkowski, "Localized transmission of electromagnetic energy," Phys. Rev. A **39**, 2005–2033 (1989).
3. I. M. Besieris, A. M. Shaarawi, and R. W. Ziolkowski, "A bidirectional travelling plane wave representation of exact solutions of the scalar wave equation," J. Math. Phys. **30**, 1254–1269 (1989).
4. A. M. Shaarawi, I. M. Besieris, and R. W. Ziolkowski, "Localized energy pulse trains launched from an open, semi-infinite, circular waveguide," J. Appl. Phys. **65**, 805–813 (1989).
5. J. Durnin, J. J. Miceli, Jr., and J. H. Eberly, "Diffraction-free beams," Phys. Rev. Lett. **58**, 1499–1501 (1987).
6. P. D. Einziger and S. Raz, "Wave solutions under complex space–time shifts," J. Opt. Soc. Am. A **4**, 3–10 (1987).
7. E. Heyman and L. B. Felsen, "Complex-source pulsed-beam fields," J. Opt. Soc. Am. A **6**, 806–817 (1989).
8. P. Hillion, "Spinor focus wave modes," J. Math. Phys. **28**, 1743–1748 (1987).
9. A. M. Shaarawi, I. M. Besieris, and R. W. Ziolkowski, "A novel approach to the synthesis of nondispersive wave packet solutions to the Klein–Gordon and the Dirac equations," J. Math. Phys. **31**, 2511–2519 (1990).
10. A. M. Vengsarkar, I. M. Besieris, A. M. Shaarawi, and R. W. Ziolkowski, "Closed-form, localized wave solutions in optical fiber waveguides," J. Opt. Soc. Am. A **9**, 937–949 (1992).
11. M. K. Tippett and R. W. Ziolkowski, "A bidirectional wave transformation of the cold plasma equations," J. Math. Phys. **32**, 488–492 (1991).
12. R. W. Ziolkowski, I. M. Besieris, and A. M. Shaarawi, "Localized wave representations of acoustic and electromagnetic radiation," Proc. IEEE **79**, 1371–1378 (1991).
13. R. Donnelly and R. W. Ziolkowski, "A method for constructing solutions of homogeneous partial differential equations: localized waves," Proc. R. Soc. London Ser. A **437**, 673–692 (1992).

14. R. W. Ziolkowski, D. K. Lewis, and B. D. Cook, "Experimental verification of the localized wave transmission effect," Phys. Rev. Lett. **62,** 147–150 (1989).
15. R. W. Ziolkowski and D. K. Lewis, "Verification of the localized wave transmission effect," J. Appl. Phys. **68,** 6083–6086 (1990).
16. E. Heyman, B. Z. Steinberg, and L. B. Felsen, "Spectral analysis of focus wave modes," J. Opt. Soc. Am. A **4,** 2081–2091 (1987).
17. E. Heyman, "Focus wave modes: a dilemma with causality," IEEE Trans. Antennas Propag. **37,** 1604–1608 (1989).
18. G. C. Sherman, A. J. Devaney, and L. Mandel, "Plane-wave expansions of the optical field," Opt. Commun. **6,** 115–118 (1972).
19. A. J. Devaney and G. C. Sherman, "Plane-wave representations for scalar wave fields," SIAM Rev. **15,** 765–786 (1973).
20. O. Yu. Zharii, "Relationship between traveling and inhomogeneous waves in the theory of transient wave processes," Sov. Phys. Acoust. **36,** 372–374 (1990) [Akust. Zh. **36,** 659–664 (1990)].
21. R. W. Ziolkowski, "Localized wave physics and engineering," Phys. Rev. A **44,** 3960–3984 (1991).
22. R. W. Ziolkowski, "Properties of electromagnetic beams generated by ultra-wide bandwidth pulse-driven arrays," Trans. IEEE Antennas Propag. **40,** 888–905 (1992).
23. P. Hillion, "Focus wave modes: remarks," J. Opt. Soc. Am. A **8,** 695 (1991).
24. D. S. Jones, *The Theory of Electromagnetism* (Pergamon, New York, 1964), pp. 38–42.
25. L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves* (Prentice-Hall, New York, 1973).
26. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).
27. J. B. Marion, *Classical Electromagnetic Radiation* (Academic, New York, 1965), Chap. 7.
28. J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 14.
29. J. A. Waak, J. H. Spencer, K. J. Johnston, and R. S. Simon, "Superluminal resupply of a stationary hot spot in 3C 395?" Astron. J. **90,** 1989–1991 (1985).
30. R. M. Bevensee, BOMA Enterprises, Alamo, California (personal communications, April 1991).
31. J. A. Kong, *Electromagnetic Wave Theory* (Wiley, New York, 1986).
32. R. Donnelly, Department of Electrical Engineering, Memorial University, St. Johns, Newfoundland (personal communication, December 1991).
33. M. Born and E. Wolf, *Principles of Optics,* 6th ed. (Pergamon, New York, 1980).