A bidirectional wave transformation of the cold plasma equations

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The bidirectional wave transformation developed for scalar equations is shown to have interesting extensions for first-order hyperbolic systems. Assuming a localized waveform of the solution gives an equation for the envelope of the localized wave. The type of the envelope equation depends on the characteristics of the original hyperbolic equations, and the speed of the localized wave. This method is applied to the cold plasma equations. In the general case integral representations are found for the fundamental solutions; and in a special case, exact solutions are constructed.

I. INTRODUCTION

There has been much interest recently in novel classes of solutions of various wave equations, the so-called localized waves. In contrast to plane waves, the localized waves are smooth solutions of hyperbolic equations, such as the wave equation, with the property that much of their energy is contained in a small, well-defined region of space-time and remains so over very large distances as they propagate from their initial position. These types of solutions are of interest for engineering applications since they offer the possibility of novel methods of energy transmission.[1] They are also of theoretical interest as possible representations of particles.[2,3]

The first examples of localized waves were Brittingham's focus wave mode solutions that were derived by heuristic arguments.[4] Ziolkowski presented a method of deriving these localized wave solutions for the wave equation.[5] He assumed the form of the solution to be

$$\Psi(x,y,z,t) = e^{i\beta(z+t)}\psi(x,y,z-t). \tag{1.1}$$

Inserting this into the scalar wave equation with wave speed normalized to unity yields a Schrödinger equation

$$[4i\beta\partial_\tau + \nabla_\perp^2]\,\psi(x,y,\tau) = 0, \tag{1.2}$$

where $\tau = z - t$, and x and y are the perpendicular coordinates. Choosing a particular solution of this equation, such as the Gaussian pulse,

$$\psi(x,y,\tau) = e^{-\beta\rho^2/(z_0+i\tau)}/4\pi i(z_0 + i\tau), \tag{1.3}$$

where $\rho = \sqrt{x^2 + y^2}$, one obtains a localized pulse solution of the wave equation that translates through space-time with only local variations:

$$\Psi(x,y,z,t) = e^{i\beta(z+t)}\frac{e^{-\beta\rho^2/(z_0+i\tau)}}{4\pi i(z_0 + i\tau)}. \tag{1.4}$$

By similar means, localized wave solutions have been found for the telegraph[6] and Klein–Gordon equations.[3]

An idea common in the derivation of these localized waves is that the solution is assumed to have a particular form. For instance, in the bidirectional form,[6] the basic solution is assumed to be a product of a forward traveling plane wave, a backward traveling plane wave, and an unknown envelope function that depends on transverse variables and one of the translating variables. For example,

$$\Psi(x,y,z,t) = \psi(x,y,\tau)e^{i\alpha\eta}e^{-i\beta\tau}, \tag{1.5}$$

where

$$\tau = \gamma(z - vt), \tag{1.6}$$

$$\eta = (z + \lambda t), \tag{1.7}$$

and α, β, γ, v, and λ are parameters. This form is then substituted into the original equation and an equation is obtained for the envelope function. The choice of parameters determines the equation for the envelope function. Since the highest-order derivatives are only affected by the parameter v, it is this parameter that determines the type of the envelope equation. In previous work the parameter v has been chosen so that the type of envelope equation is either elliptic or parabolic. The choice of the other parameters, while not affecting the type of the envelope equation, is also important. For example, if the resulting envelope equation is elliptic, the existence of regular solutions vanishing at infinity is dependent on the undifferentiated terms in the equation.

Using the results for scalar equations, results have been obtained for systems of equations. For example, localized traveling wave solutions to Maxwell's equations have been presented using solutions to the scalar wave equation and Hertz potentials.[1] Also, solutions to the Dirac equations were found using solutions of the Klein–Gordon equation.[3]

The equations encountered in mathematical physics for wave propagation are often in the form of a system of linear first-order hyperbolic partial differential equations, which, in general, cannot be reduced to a single scalar equation. Thus it is of interest to know if such localized wave solutions can be found for these problems. In this paper, the methods for finding localized wave solutions for scalar equations are extended to first-order linear systems of hyperbolic equations. The form of the solution described above is assumed and a new system is obtained. It is desirable that the type of the new system be known; we give a simple criteria that ensures the resulting system be elliptic or parabolic.

The method is then applied to a specific system of equations, the cold plasma equations. For a special case (an unmagnetized plasma) the cold plasma equations can be brought into a vector Klein–Gordon equation form and the solutions obtained directly. However, in general, no such reduction is apparent. In this case the equations can be transformed to an elliptic system and a fundamental solution found by the method of Fourier transform.

II. TRANSFORMATION OF FIRST-ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Consider the linear first-order system of partial differential equations in one time and three space variables:

$$u_t + A_1 u_x + A_2 u_y + A_3 u_z + A_4 u = 0, \tag{2.1}$$

where u is an n vector, and the A_i are $n\times n$ matrices, possibly functions of (x,y,z,t). The type of the equation system is determined by the existence of real characteristics or equivalently, the existence of propagating wave fronts. The equation for the characteristics is

$$P(\phi_t,\phi_x,\phi_y,\phi_z) = \det(\phi_t I + \phi_x A_1 + \phi_y A_2 + \phi_z A_3) = 0. \tag{2.2}$$

The equations are hyperbolic if for any ϕ_x,ϕ_y,ϕ_z not identically equal to zero, the characteristic equation $P(\phi_t,\phi_x,\phi_y,\phi_z) = 0$ has n real roots counting multiplicities. The speed s at which a disturbance propagates is given by

$$s = |\phi_t|/\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}. \tag{2.3}$$

We add the following assumption: that if $P(\phi_t,\phi_x,\phi_y,\phi_z) = 0$, then

$$\min|\phi_t/\phi_z| = S > 0. \tag{2.4}$$

This means that we will assume that all disturbances propagate in the z direction with finite speed strictly greater than zero. This assumption is violated if there are time independent propagating disturbances, e.g., static solutions permit infinite propagation speeds.

Now, as discussed above, assume the solution to be a product of backward and forward traveling waves with a function depending on the transverse variables (x,y) and the translating variable τ, that is,

$$u(x,y,z,t) = u(x,y,\tau)e^{i\alpha\eta}e^{-i\beta\tau}, \tag{2.5}$$

where the translating variables

$$\tau = \gamma(z - vt), \tag{2.6}$$

$$\eta = (z + \lambda t). \tag{2.7}$$

The constants α and β have dimensions of inverse length and the constants λ and v have dimensions of velocity so that τ and η are spatial variables. Therefore, we are trying to construct a localized wave propagating in the z direction. Inserting the form (2.5) into the original equation yields a first-order system with independent variables x, y, and τ:

$$\gamma(A_3 - vI)u_\tau + A_1 u_x + A_2 u_y + \tilde{A}_4 u = 0, \tag{2.8}$$

where

$$\tilde{A}_4 = A_4 + i(\alpha - \beta\gamma)A_3 + i(\alpha\lambda - \beta v)I. \tag{2.9}$$

The equation for the characteristics of this system is

$$\mathrm{Det}(\phi_\tau \gamma(A_3 - vI) + \phi_x A_1 + \phi_y A_2) = 0. \tag{2.10}$$

This equation can be written in terms of the polynomial P defined by the original system. In particular, the characteristic condition becomes

$$P(-v\gamma\phi_\tau,\phi_x,\phi_y,\gamma\phi_\tau) = 0. \tag{2.11}$$

Using the assumption that the original system has a minimum speed, $S > 0$, at which disturbances propagate in the z direction, one sees that the characteristic equation

$$P(a,b,c,d,e) = 0 \tag{2.12}$$

has real roots only when

$$|a/e| \geqslant S. \tag{2.13}$$

Thus the characteristic equation has no real roots if $|v| \leqslant S$; that is, the transformed system is elliptic if the solution translates at a speed less than the slowest speed of propagation in the z direction of the original system. The distinction between the slowest speed of propagation of the system and the slowest speed of propagation in the z direction is needed only for systems where there is a preferred direction. For isotropic equations such as the wave equation, the choice of the direction of propagation of the localized wave is arbitrary. However, for a system such as the magnetohydrodynamics equations where wave propagation is highly directional, the choice of direction of propagation of the localized wave depends on the wave structure of the system.

This condition for ellipticity is quite natural. In a hyperbolic system the speed at which information propagates through the system is limited. Linear combinations of the dependent variables propagate at different speeds. In the present case, by forcing the data to move at a single speed v, the speed at which the data propagates is limited by the slowest speed of propagation in the z direction of the system. On the other hand, if v is chosen to be one of the eigenvalues of A_3, as in the case of the focus wave modes, the transformed system is parabolic in the sense that one cannot solve for the τ derivatives. This choice of v corresponds to constructing a localized wave solution propagating at the same speed at which disturbances propagate in the z direction. Solutions to either elliptic or parabolic problems may typically have some weak decay properties at infinity so that the solution to the original problem with either of these choices for v will be a localized modulated traveling wave.

Note that as in the scalar case the traveling plane waves play no role in determining the type of the system. However, they are important in determining the nature of the solutions to the elliptic problem because they produce the undifferentiated terms contained in $\tilde{A}_4$.

If the original problem has constant coefficients, a first step in understanding the resulting elliptic problem is to find a fundamental solution for the elliptic problem using Fourier transform methods. This is the approach we have taken with the cold plasma equations.

III. LINEARIZED COLD PLASMA EQUATIONS

We would now like to apply the method of Sec. II to the linearized cold plasma equations as found, for example in Ref. 7:

$$v_t = (q/m)[E + (1/c)v\times B_0], \tag{3.1a}$$

$$c\nabla\times B = 4\pi n_0 q v + E_t, \tag{3.1b}$$

$$c\nabla\times E = -B_t, \tag{3.1c}$$

with $B_0 = B_0\hat{z}$ being the background magnetic field, B the magnetic field vector, E the electric field strength vector, v the velocity vector, q the charge of an electron, m its mass, and c the speed of light. Momentum balance is represented by Eqs. (3.1a), and Eqs. (3.1b) and (3.1c) are the usual Maxwell equations. We take the plasma to be an electron plasma, neglecting the heavier ions. These equations are of interest because they model wave propagation in a plasma such as the one found in the upper atmosphere, where thermal effects can be neglected. Setting the vector

$$u = \begin{pmatrix} v \\ E \\ B \end{pmatrix}, \tag{3.2}$$

we can write the cold plasma equations in the form of (2.1), a first-order system of nine partial differential equations. We will use the same notation as in Sec. II for the coefficient matrices A_i, which may be trivially written down. Note that the coefficient matrices are constants. The equation for the characteristics as given by (2.2) is

$$\phi_t^5(\phi_t^2/c^2 - \phi_x^2 - \phi_y^2 - \phi_z^2)^2 = 0, \tag{3.3}$$

which shows that the equations are indeed hyperbolic. Clearly, these equations are highly degenerate; and this degeneracy is seen in the leading factor: ϕ_t^5. Because of this factor, the assumption in (2.4) does not hold. These degeneracies come from the existence of time independent disturbances (static solutions) of the form

$$\begin{aligned} E &= \nabla\Phi(x,y,z), \\ B &= \nabla\Psi(x,y,z), \\ v &= v(x,y,z), \end{aligned} \tag{3.4}$$

in which there are five arbitrary functions. However, standard techniques exist that can be used to deal with this degeneracy.

Invoking the bidirectional solution form (2.5), results in a new first-order system with independent variables x,y,τ as in (2.8). The characteristic equation for the transformed plasma equation system is given by (2.9) to be

$$-(\gamma\nu\phi_\tau)^5((c^2 - \nu^2)(\gamma\phi_\tau)^2 + c^2\phi_x^2 + c^2\phi_y^2)^2 = 0. \tag{3.5}$$

Due to the nature of the original equations (3.1), the new system is only partially elliptic if $\nu < c$. As stated before, if ν is chosen to be one of the eigenvalues of A_3: c, $-c$, or 0, one cannot solve for the τ derivatives and the system is parabolic.

A first step in understanding the system of transformed equations is to find its fundamental solution or Green's function. This fundamental solution is obtained by solving the corresponding inhomogeneous system having a forcing delta function on its right-hand side. This procedure can be accomplished, at least formally, using the method of Fourier transform. The Fourier transform variables are

$$(\tau,x,y) \to (k_\tau,k_x,k_y).$$

Using the familiar properties of the Fourier transform we have an algebraic equation for $\hat{u}$, the Fourier transform of u:

$$W\hat{u} = 1, \tag{3.6}$$

where W is explicitly

$$W = \gamma(A_3 - \nu I)k_\tau + A_1 k_x + A_2 k_y + \widetilde{A}_4, \tag{3.7}$$

and 1 is a vector of ones. The matrix W can be explicitly inverted in (3.6) to define the solution vector in Fourier space:

$$\hat{u} = W^{-1}\cdot 1 = (\widetilde{W}\cdot 1)/Q(k_\tau,k_x,k_y), \tag{3.8}$$

where Q is the determinant of W and hence a polynomial in k_τ, k_x, and k_y, and $\widetilde{W}$ is defined by (3.8). In fact, Q is a polynomial in k_τ and $k_\perp^2$ where the transverse wave number

$$k_\perp = \sqrt{k_x^2 + k_y^2}. \tag{3.9}$$

The form of the denominator gives information as to the structure of the solutions. The polynomial $Q(k_\tau,k_\perp)$ is of ninth degree in k_τ and of fourth degree in $k_\perp$ with only even powers of $k_\perp$ appearing. The final step is to invert the transforms. Using the method of residues, one can invert (in principle) the transforms in k_τ and $k_\perp$. Because Q is fourth degree in $k_\perp$ with only even powers appearing, one can always find the roots in $k_\perp$ and then invert that transform. In some special cases Q can be factored explicitly and both the inverse transforms in $k_\perp$ and k_τ can be computed.

We introduce the standard definitions for the cyclotron frequency and the plasma frequency, respectively:

$$\omega_p^2 = 4\pi n_0 q^2/m, \tag{3.10a}$$

$$\omega_{ce} = qB_0/mc. \tag{3.10b}$$

In the special case where $\alpha = \beta = 0$, $\nu = c$, and $\gamma = 1$, the transformed solution reduces to the form:

$$\hat{u} = (\widetilde{W}\cdot 1)/k_\tau^5(k_\tau^2c^2 - a^2)^2(k_\perp^2c^2 + \omega_p^2)^2, \tag{3.11}$$

where

$$a^2 = \omega_{ce}^2 k_\perp^4 c^4/(k_\perp^2c^2 + \omega_p^2)^2 + \omega_p^2. \tag{3.12}$$

Another special case is $B_0 = 0$, $\alpha = \beta = 0$, γ and ν arbitrary. The transformed solution in this case becomes:

$$\hat{u} = \frac{\widetilde{W}\cdot 1}{(\gamma\nu k_\tau)^3((\gamma\nu k_\tau)^2 - \omega_p^2)(-c^2k_\perp^2 + (\nu^2 - c^2)k_\tau^2 - \omega_p^2)^2}. \tag{3.13}$$

In these special cases we can formally invert the transforms in the k_τ and $k_\perp$ variables, leaving only an integral with respect to $\theta = \tan^{-1}(k_y/k_x)$. Consequently, an integral representation of the fundamental localized wave solution can be constructed from which asymptotic information could be found.

For the special case of $v\times B_0 = 0$ some solutions can be written down directly. First, the equation system can be combined to obtain a vector Klein–Gordon equation. In particular, combining Eqs. (3.1b) and (3.1c), we obtain

$$c^2\nabla\times\nabla\times B = n_0 q\nabla\times v - B_{tt}$$

or

$$(\partial_{tt} - c^2\nabla^2)B = n_0 q\nabla\times v, \tag{3.14}$$

since $\nabla\cdot B = 0$. Taking the derivative with respect to t of (3.14) yields

$$(\partial_{tt} - c^2\nabla^2 + \omega_p^2)B_t = 0. \tag{3.15}$$

We now examine the envelope equation. Assuming the bidirectional solution form (2.5),

$$B_t = \widetilde{B}_t(x,y,\tau)e^{i\alpha\eta}e^{-i\beta\tau}, \tag{3.16}$$

the equation for the forward propagating envelope $\widetilde{B}_t$ is

$$[-c^2\nabla_\perp^2 - \gamma^2(c^2 - v^2)\partial_\tau^2 + 2ia\partial_\tau + b]\widetilde{B}_t = 0, \tag{3.17}$$

where

$$a = \gamma^2(c^2 - v^2)\beta - \alpha\gamma(c^2 + \lambda v), \tag{3.18a}$$

$$b = \alpha^2(c^2 - \lambda^2) + \beta^2\gamma^2(c^2 - v^2) - 2\alpha\beta\gamma(c^2 + \lambda v) + \omega_p^2, \tag{3.18b}$$

$$\nabla_\perp^2 = \partial_x^2 + \partial_y^2. \tag{3.18c}$$

For $v < c$ Eq. (3.17) is elliptic. By an appropriate choice of constants this operator can be made either a Schrödinger operator or a Helmholtz operator. To obtain a Schrödinger operator, setting $v = \lambda = c$, $\beta = 0$ gives

$$(c^2\nabla_\perp^2 + 4i\alpha\gamma\partial_\tau)\widetilde{B}_t = 0. \tag{3.19}$$

To get a Helmholtz operator, we set

$$\gamma^2 = c^2/(c^2 - v^2). \tag{3.20}$$

This gives

$$(-c^2\widetilde{\nabla}^2 + 2ia'\partial_\tau + b')\widetilde{B}_t = 0, \tag{3.21}$$

where

$$\widetilde{\nabla}^2 = \nabla_\perp^2 + \partial_\tau^2, \tag{3.22a}$$

$$a' = c^2\beta - \alpha\gamma(c^2 + \lambda v), \tag{3.22b}$$

$$b' = \alpha^2(c^2 - \lambda^2) + c^2\beta^2 - 2\alpha\beta\gamma(c^2 + \lambda v) + \omega_p^2. \tag{3.22c}$$

To eliminate the first derivatives we set

$$\lambda = (\beta/\alpha\gamma - 1)(c^2/v). \tag{3.23}$$

This gives

$$[-c^2\widetilde{\nabla}^2 + \alpha^2(c^2 - \lambda^2) - \beta^2c^2 + \omega_p^2]\widetilde{B}_t = 0. \tag{3.24}$$

For there to be nonzero, nonsingular solutions of this equation, which vanish at infinity, one must have

$$\alpha^2(c^2 - \lambda^2) - \beta^2c^2 + \omega_p^2 < 0. \tag{3.25}$$

In Ref. 3 the choice

$$\lambda = c^2(1/v) \tag{3.26a}$$

was made which implies

$$\beta = 2\alpha\gamma. \tag{3.26b}$$

In this case, the condition (3.25) becomes

$$-4\alpha^2\gamma^2c^2 - \alpha^2c^4/\gamma^2v^2 + \omega_p^2 < 0. \tag{3.27}$$

Choosing

$$v^2 = \alpha^2c^4/\omega_p^2\gamma^2, \tag{3.28}$$

which requires

$$\alpha = \gamma(v/c)\omega_p/c, \tag{3.29}$$

further reduces this condition to the simple form:

$$-4\alpha^2\gamma^2c^2 < 0. \tag{3.30}$$

These choices of parameters result in a Helmholtz equation for the envelope solution:

$$(\widetilde{\nabla}^2 + 4\gamma^2\alpha^2)\widetilde{B}_t = 0. \tag{3.31}$$

A general solution to this equation is

$$\psi(x,y,\tau) = j_l(2\alpha\gamma R)P_l^m(\tau/R)\cos(m\theta), \tag{3.32}$$

where $R = \sqrt{x^2 + y^2 + \tau^2}$, j_l is the spherical Bessel function of order l, and P_l^m is the associated Legendre function. As noted in Ref. 3, although the integral of $|\psi|^2$ is not finite, finite energy solutions can be constructed by taking a superposition of the fundamental solutions (1.5) with $\psi(x,y,\tau)$ given by (3.32).

Using the localized solutions of the Klein–Gordon equation derived in Ref. 6, solutions can be constructed to the cold plasma equations (3.1). To form a solution that is consistent with the assumption that $v\times B_0 = 0$, either v is parallel to B_0 or $B_0 = 0$. For the case of v parallel to B_0, a localized wave solution of the cold plasma equations can then be constructed with (3.32) as follows:

$$A = \hat{z}\Psi, \tag{3.33a}$$

$$B = \nabla\times A, \tag{3.33b}$$

$$E = -A_t - \nabla\Phi_t, \tag{3.33c}$$

$$v = -(q/m)A - \nabla\Phi, \tag{3.33d}$$

where

$$\Psi(x,y,z,t) = \psi(x,y,\tau)e^{i\alpha\eta}e^{-i\beta\tau} \tag{3.34}$$

is a localized wave solution to the Klein–Gordon equation (3.15) and the terms A and Φ are constrained by the relation:

$$\nabla(\Phi_{tt} + \omega_p^2\Phi - \nabla\cdot A) = 0. \tag{3.35}$$

Note that the role played by A and Φ_t in Eqs. (3.33b) and (3.31c) is like that of the usual vector and scalar potentials in electromagnetic theory. Equation (3.35) can be understood then as specifying a gauge. A specific choice of Φ is to choose

$$\Phi = \frac{1}{\omega_p}\int^t \nabla\cdot A(s)\sin(\omega_p(t-s))ds. \tag{3.36}$$

Note that the flow is a compressible one and that the divergence of the electric field is nonzero and localized, indicating some moving localized charge density.

For the case $B_0 = 0$ we construct a different type of localized wave solution by choosing

$$A = \nabla\times(\hat{z}\Psi) \tag{3.37}$$

and choosing E, B, and v as before. Since $\nabla\cdot A = 0$ we may chose $\Phi = 0$. This gives a solution where, unlike the solution described in (3.36), the flow is incompressible and the divergence of E is zero.

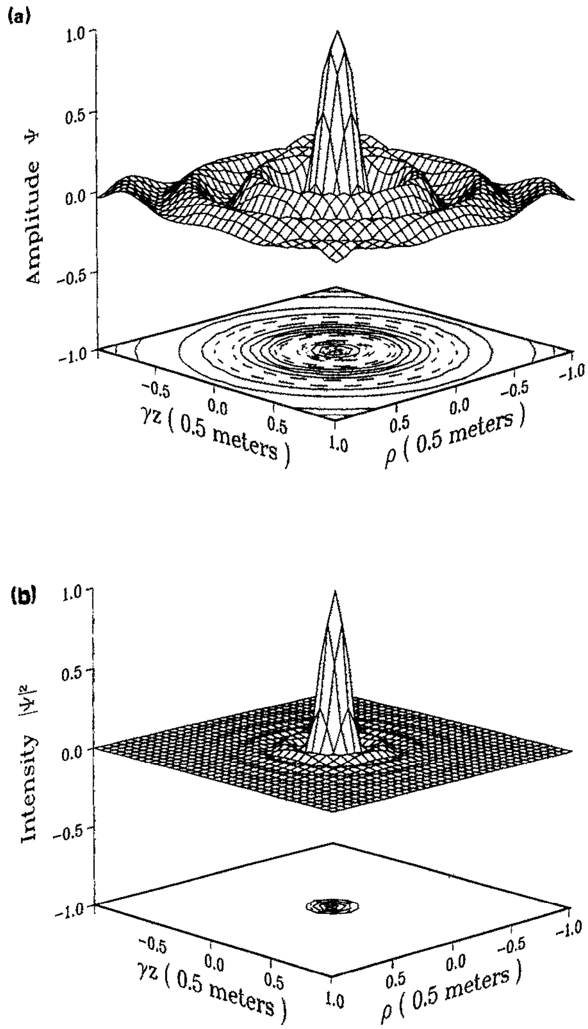

FIG. 1. The localized wave solution Ψ is shown as a function of the spatial variables at $t = 0$. (a) Its amplitude Ψ and (b) its energy density $|\Psi|^2$.

In Fig. 1, plots are shown of Ψ and $|\Psi|^2$ for parameters typical of those found in the upper atmosphere. For simplicity we choose a radially symmetric solution with $l = m = 0$. The only parameter entering the solution is the number density which we take as $n_0 = 3\times 10^4\,\mathrm{cm}^{-3}$. This gives a plasma frequency: $\omega_p = 9.77\times 10^6$ rad/s. The parameters α and γ are related through (3.18) and (3.26), respectively, to the group velocity v which we take to be $v = 0.999c$. From (3.18) we find that $\gamma = 22.366$, and from (3.26) that $\alpha = 7.2808\times 10^{-3}\,\mathrm{cm}^{-1}$. The approximate "size" of the localized solution can be taken as the first zero of the spherical Bessel function times 3.0704 cm. Note that the localization in the z direction is scaled by γ.

IV. CONCLUSIONS

The bidirectional wave transformation developed for scalar equations was extended to first-order hyperbolic systems. Assuming a localized waveform of the solution gives an equation for the envelope of the localized wave. The type of the envelope equation depends on the characteristics of the original hyperbolic equations, and the speed of the localized wave. In particular, if the speed of the localized wave is less than the speeds associated with the characteristics of the original equations, the envelope equation is elliptic.

This method was applied to the cold plasma equations. In the general case, integral representations were found for the fundamental solutions; and in a special case, exact solutions were constructed. These exact solutions were of two types, one compressible, the other incompressible. An unanswered question, however, is the possibility of exciting such solutions. This issue is beyond the scope of the present work and will be addressed in future efforts.

The existence of these localized waves is of interest both for the engineering possibilities they suggest and as a way of understanding wave propagation in linear hyperbolic partial differential equations. In this paper we have made explicit the relation between localized wave solutions and the wave structure of the differential equation. This characterization suggests that solutions of this type may indeed be typical of linear hyperbolic equations. There are many other equation sets such as the equations of magnetohydrodynamics for which the existence of localized waves would be of interest. Moreover, this decomposition offers novel ways of analyzing the wave structure of linear hyperbolic equations. For example, the analysis done in scattering theory using plane waves as a solution basis might be reexamined using localized waves.

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