On the evanescent fields and the causality of the focus wave modes

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The diverging and converging field components of the source-free focus wave modes are studied within the framework of both the Whittaker and Weyl plane wave expansions. It is shown that, in the Weyl picture, the evanescent fields associated with the diverging and converging components of the focus wave mode solution cancel each other identically. The source-free focus wave modes are, hence, composed solely of backward and forward propagating components of the Whittaker type. It will also be shown that no evanescent fields are associated with the causal excitation of an aperture by an initial focus wave mode field. The diverging field, however, is composed solely of causal components that propagate away from the aperture. With a specific choice of parameters, the field generated by the aperture is a very good approximation to the source-free solution. Under the same conditions, the original focus wave mode solution is composed predominantly of causal forward propagating fields. © *1995 American Institute of Physics.*

I. INTRODUCTION

A full decade has passed since the focus wave modes (FWM) solution to the Maxwell equations has been introduced.[1] Similar solutions have been derived for the scalar wave equation, as well as for other wave equations.[2–8] The FWM is a nonsingular smooth Gaussian pulse-like solution that is characterized by having an infinite total energy content. This last property renders the FWM field to be physically unrealizable. To circumvent such a difficulty, it has been suggested that a superposition of the FWM can produce finite-energy pulses of a large bandwidth that can exhibit extended ranges of localization. A large number of such localized wave (LW) solutions have been reported recently.[2–13] The ability of such pulses to display desirable localized transmission characteristics has been demonstrated, both theoretically and experimentally. It has also been shown that such fields can be excited from finite apertures.[14] The resulting fields approximate to a great extent the theoretically predicted solutions. Experiments[15,16] verifying the propagation of such localized waves utilized independently addressable, finite-sized arrays.

It is well known that the aforementioned localized wave solutions are composed of forward and backward propagating components. Such a behavior is reminiscent of the original source-free FWM solution, which has been criticized for being dominated by backward propagating components.[17,18] It has also been argued that this causes "grave" problems with causality and the possibility of launching such LW solutions. Nonetheless, the analysis adopted in the aforementioned study on the causality of the FWM has been restricted to a special case.[18] In the present work, it will be demonstrated that under a different condition (not considered by Ref. 18) the FWM solution is composed predominantly of forward propagating components. Furthermore, we

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shall consider two distinct situations, the original source-free FWM solution and the forward illumination of an infinite aperture by an initial FWM field. It will be shown that under special conditions, the causal field of the FWM aperture is a very good approximation to the source-free FWM. In principle, the same ideas apply directly to other LW solutions. Nevertheless, the mathematics is more involved and will not be carried out here.

It should be pointed out, however, that an aperture excited by an initial FWM field has a finite size for all times except at $t=-\infty$ and $+\infty$; i.e., at its initial and final moments of excitation. The effective radius of the aperture decreases from an infinite radius at $t=-\infty$ to a finite value at $t=0$, and then the aperture expands to reach once more an infinite size at $t=+\infty$. The intensity of the field illuminating the FWM aperture decreases to zero at the same rate as that of the expansion of its area, as the aperture grows toward an infinite size. These two effects balance each other and the power of the field illuminating the aperture remains constant and finite for all times. However, the excitation of the FWM aperture utilizes infinite energy *only* because it needs to be illuminated for an infinitely long time. On the other hand, the excitation of the FWM aperture does not need infinite power, in contradistinction to other excitation modes of infinite apertures, e.g., the Bessel beams and the plane wave illuminations. Thus, as far as we are concerned, there are no problems *per se* with the excitation of the FWM field, except for the need of an infinite time to illuminate its aperture. Hence, a FWM aperture excited for a very long but a finite period of time is expected to produce a field that is a good approximation to the original FWM solution. Such an idea is out of the scope of this paper, and is pursued further in another work.

It has also been pointed out[19] that the source-free FWM solution does not contain any evanescent wave components. Most of the aforementioned LW solutions are, however, finite-energy superpositions of the original focus wave modes. One should then expect that the finite-energy LW solutions, composed as a superposition of the source-free FWM, have no evanescent waves associated with them. The best mathematical framework to approach such issues of causality and evanescent fields is to use the angular spectrum superposition.[20] In the various sections of this work we shall start with the more familiar Fourier composition and then introduce the angular variables for which the analysis becomes more transparent. Traditionally, solutions to the three-dimensional scalar wave equation can be derived in two distinct fashions. One is due to Whittaker,[21] in which he uses a superposition of homogeneous plane waves propagating in opposite directions. The other has been used by Weyl[21] to express the fields outside the source region as a combination of propagating homogeneous plane waves, together with the associated inhomogeneous evanescent waves. There have been several attempts to further our understanding of the relationship between these two distinct representations.[14,22–24] It has been claimed that the portion of the field represented as an expansion of the homogeneous backward propagating plane waves is equivalent to that expressed as a superposition of the inhomogeneous evanescent modes.[14,23] The physical meaning of such equivalence and its temporal and spatial domains of validity are not fully comprehended. It is our aim in this work to further our understanding of the Whittaker and Weyl representations by considering both the source-free FWM solution and the FWM illuminated aperture. The FWM field provides a rich example for such an investigation because of its unusual forward, backward propagating, and evanescent structure.

In this work, we try to use both approaches to rederive the FWM solution to the scalar wave equation. Most of the calculations will be carried in the angular spectrum representation. It will be shown that the source-free FWM does not have any evanescent fields associated with it, because the diverging and converging inhomogeneous Weyl components cancel out identically. On the other hand, the FWM aperture does not exhibit any evanescent fields for completely different reasons. Specifically, the cancellation of the evanescent components follows from a condition related to the causal forward illumination of the aperture.

The plan of this work is to study the angular spectrum representation for the source-free FWM in both the Whittaker and the Weyl representations. The annulment of its evanescent field is expounded upon and the predominance of the forward propagating components under specific conditions is demonstrated. This shows that, contrary to what has been claimed[18] for a specific choice of parameters, the source-free FWM can be composed primarily of forward propagating fields if a different set of parameters is chosen. Furthermore, it is shown that the outgoing forward illumination of the FWM aperture produces a diverging causal field (i.e., a field that does not contain any acausal components), which resembles to a great extent the source-free FWM solution. We, thus, establish the possibility of exciting fields that approximate the original focus wave modes, and we can, consequently, claim that there is nothing "grave" about the causality of the LW solutions in general.

II. THE ANGULAR SPECTRUM OF THE SOURCE-FREE FOCUS WAVE MODES

In this section, we shall deal with the source-free FWM. The diverging, converging, and evanescent components of such a solution will be all derived explicitly. For future comparisons between the spectral content of such components, we choose to work with the azimuthally symmetric normalized FWM solution to the 3-D scalar wave equation; specifically,

$$\Psi_{sf}(\mathbf{r},t)=\frac{a_1}{4\pi(a_1+i(z-ct))}\, e^{-\beta\rho^2/(a_1+i(z-ct))}e^{i\beta(z+ct)}. \tag{2.1}$$

Such a pulse-like solution has an amplitude equal to $1/(4\pi)$ at its center. The source-free FWM can be synthesized from a bidirectional representation[3] that provides the most suitable basis for such a wave solution. A simple transformation links the bidirectional representation to the Fourier one. In the Fourier picture, the Fourier superposition leading to the FWM pulse solution[3,8] can be written as

$$\begin{aligned}\Psi_{sf}(\mathbf{r},t)=\frac{a_1}{(2\pi)^2}\int_0^\infty d\chi\int_0^\infty d\omega\int_{-\infty}^\infty dk_z\,\frac{\pi}{4c}\,e^{-a_1(k_z+(\omega/c))/2}\delta\!\left(\left(\frac{\omega}{2c}\right)-\left(\frac{k_z}{2}\right)-\beta\right)\\ \times\chi J_0(\chi\rho)e^{-ik_zz}e^{+i\omega t}\delta((\omega/2c)^2-(k_z/2)^2-(\chi/2)^2).\end{aligned} \tag{2.2}$$

The above integration may be carried out in two distinct fashions, each leading to a different representation. In particular, one can integrate over ω first, thus, ending up with a Whittaker type of expansion. In contradistinction, an integration over k_z first leads to the Weyl superposition over homogeneous and inhomogeneous plane waves. Using the former approach, we integrate over ω first to obtain

$$\Psi_{sf}(\mathbf{r},t)=\frac{a_1}{(2\pi)^2}\int_0^\infty d\chi\int_{-\infty}^\infty dk_z\,\frac{\pi c}{\omega_+}\,e^{-a_1(k_z+(\omega_+/c))/2}\delta(\omega_+-k_z-2\beta)\chi J_0(\chi\rho)e^{-ik_zz}e^{+i\omega_+t}. \tag{2.3}$$

Here $\omega_+=c\sqrt{k_z^2+\chi^2}$. The integration over k_z can be divided into two components. One is traveling in the positive z direction, while the other is propagating in the negative z direction. In order to separate the forward and backward propagating components, it is more natural to transform the above integration to the angular spectrum representation. The angular spectral content is usually expressed as a superposition over the spherical angles α and β of the propagation vector $\mathbf{k}$.[20] The Fourier spectrum of the FWM is azimuthally symmetric [cf. Eq. (2.1)], hence, the corresponding angular spectrum is independent of β. We introduce the new angular variables,

$$\chi=\kappa\sin\alpha \quad\text{and}\quad k_z=\kappa\cos\alpha, \quad\text{with}\quad \omega_+=\kappa c. \tag{2.4}$$

Such a change of variables transforms the integral in Eq. (2.3) into the following form:

$$\Psi_{sf}(\mathbf{r},t)=\frac{a_1}{4\pi}\int_0^{\pi}d\alpha\int_0^{\infty}d\kappa\ \kappa\ \sin\ \alpha\ J_0(\kappa\rho\ \sin\ \alpha)$$
$$\times\,\delta(\kappa-\kappa\ \cos\ \alpha-2\beta)e^{-\kappa a_1(\cos\alpha+1)/2}e^{-i\kappa(z\cos\alpha-ct)}. \tag{2.5}$$

The limits of the integration over α ranges from 0 to π. Notice that cos α takes positive (negative) values for $0\leqslant\alpha<\pi/2$ ($\pi/2<\alpha\leqslant\pi$), thus resulting in a superposition over plane waves traveling in the positive (negative) z direction. The angular superposition given in Eq. (2.5) can, thus, be split into the following forward and backward Whittaker propagating components,

$$\Psi_{sf}(\mathbf{r},t)=\Psi^{(+)}(\mathbf{r},t)+\Psi^{(-)}(\mathbf{r},t). \tag{2.6a}$$

The positive and negative traveling components are given explicitly in terms of the integration,

$$\Psi(\mathbf{r},t;a,b)=\frac{a_1}{4\pi}\int_a^{b}d\alpha\int_0^{\infty}d\kappa\ \kappa\ \sin\ \alpha\ J_0(\kappa\rho\ \sin\ \alpha)$$
$$\times\,\delta(\kappa-\kappa\ \cos\ \alpha-2\beta)e^{-\kappa a_1(\cos\alpha+1)/2}e^{-i\kappa(z\cos\alpha-ct)}, \tag{2.6b}$$

where $\Psi^{(+)}(\mathbf{r},t)=\Psi(\mathbf{r},t;0,\pi/2)$ and $\Psi^{(-)}(\mathbf{r},t)=\Psi(\mathbf{r},t;\pi/2,\pi)$. The integration over κ can be carried out to yield

$$\Psi(\mathbf{r},t;a,b)=\frac{1}{2\pi}\int_a^{b}d\alpha\ \mathscr{A}(\alpha)J_0\left(\frac{2\beta\rho\ \sin\ \alpha}{(1-\cos\ \alpha)}\right)e^{-i2\beta(z\cos\alpha-ct)/(1-\cos\alpha)}. \tag{2.7a}$$

Notice that the integral representations of $\Psi^{(+)}(\mathbf{r},t)$ and $\Psi^{(-)}(\mathbf{r},t)$ share the same spectrum, but differ only by their limits of integration. The spectral content $\mathscr{A}(\alpha)$ is given explicitly as

$$\mathscr{A}(\alpha)=\frac{\beta a_1\ \sin\ \alpha}{(1-\cos\ \alpha)^2}\,e^{\beta a_1}e^{-2\beta a_1/(1-\cos\alpha)}. \tag{2.7b}$$

Such a function is graphed in Figs. 1(a) and 1(b) for different values of βa_1. The ranges of α contributing to the forward and backward components are specified on the graphs. It will be shown in a later section, that to launch the FWM field from an aperture situated at the $z=0$ plane, only the forward components can be causally propagated in the positive z direction. In view of such an excitation scheme, the Whittaker forward propagating components are perceived as being causal, while the backward traveling fields are considered acausal because they cannot be propagated forward in a causal manner. It is clear from Fig. 1(a) that the forward spectral components are dominant when $\beta a_1<1$, and for all practical purposes *almost* all of the spectrum is found in the causal range when $\beta a_1\ll 1$. The opposite is true when $\beta a_1>1$, which is the case considered by Heyman.[18] One can see from Fig. 1(b) that all the spectral components contribute to the backward field when $\beta a_1\gg 1$. This supports the conclusions of Ref. 18 under that specific choice of parameters. As for the former case ($\beta a_1\ll 1$), which has not been discussed in Ref. 18, the situation is completely different. A detailed discussion of the differences between the two cases is deferred to the next section. As for now, we need to stress that unlike the Fourier superposition the integration over α is of a finite range. We do not have to worry about tails of small amplitudes contributing significantly by integrating their values over an infinite domain. For the angular spectrum, the forward (causal) and backward (acausal) ranges are equal, subsequently, the portion having larger spectral amplitudes contributes more significantly to the total FWM field.

The forward and backward propagating fields of the Whittaker representation can be cast in another form by introducing the new variable $\lambda=\cot(\alpha/2)$, which reduces Eq. (2.7) to

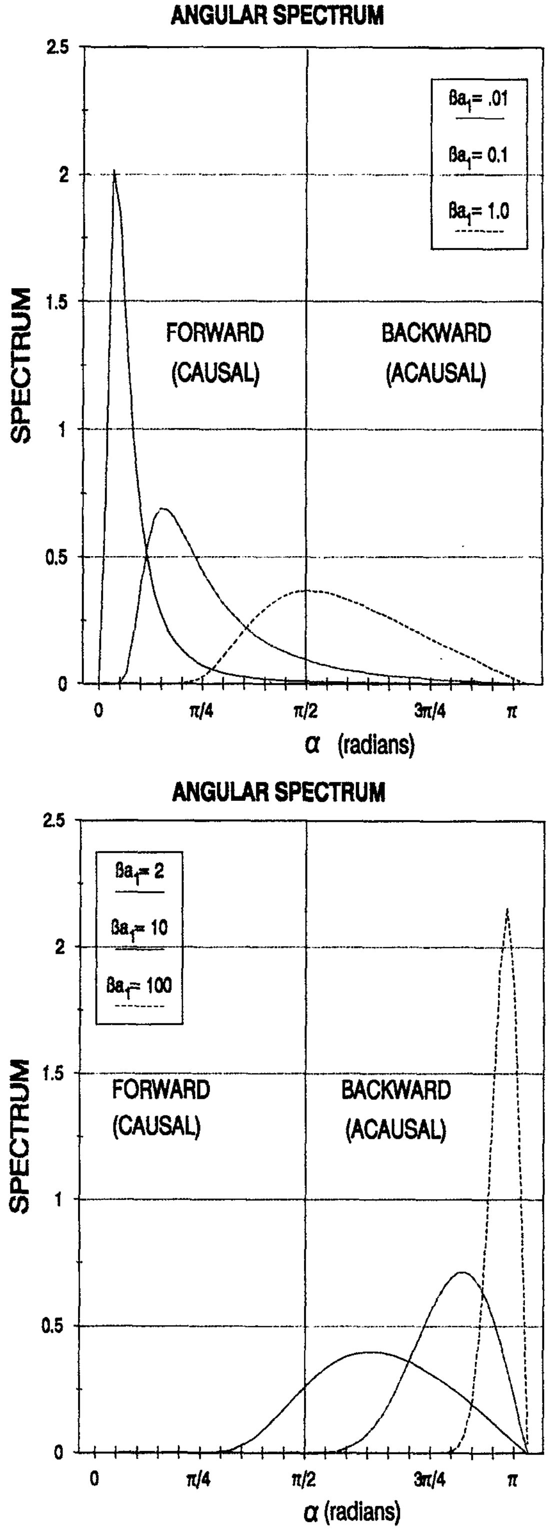

FIG. 1. The angular spectral content for the source-free FWM solution when (a) $\beta a_1 \leq 1$ and (b) $\beta a_1 > 1$ plotted for various values of the βa_1 parameter.

$$\Psi(\mathbf{r},t;a',b') = \frac{\beta a_1}{2\pi} \int_{a'}^{b'} d\lambda \; \lambda J_0(2\beta\rho\lambda) e^{-\beta[a_1+i(z-ct)]\lambda^2} e^{i\beta(z+ct)}, \tag{2.8a}$$

where the forward and backward components are given by

$$\Psi^{(+)}(\mathbf{r},t) = \Psi(\mathbf{r},t;1,\infty) \quad \text{and} \quad \Psi^{(-)}(\mathbf{r},t) = \Psi(\mathbf{r},t;0,1). \tag{2.8b}$$

Such a representation will be useful when comparing the relative strength of both components in the asymptotic analysis of the next section. It also helps in understanding some of the more subtle aspects of the Weyl representation and the annulment of the associated evanescent fields. When these two integrations are summed up, the integration over the limits from 0 to ∞ can be evaluated[3] to give the source-free FWM solution given in Eq. (2.1).

Next, we consider the Weyl representation by integrating Eq. (2.2) over the k_z variable first. Since k_z can have both positive and negative values, the integration may be split into two parts, viz.,

$$\Psi_{\rm sf}(\mathbf{r},t) = \Psi^{(d)}(\mathbf{r},t) + \Psi^{(c)}(\mathbf{r},t), \tag{2.9a}$$

where $\Psi^{(d)}(\mathbf{r},t)$ and $\Psi^{(c)}(\mathbf{r},t)$ correspond to waves diverging and converging on an aperture situated at $z=0$. In what follows, we shall be only interested in the Weyl expansion associated with the positive z half-space. The two components in Eq. (2.9a) can be written explicitly as

$$\Psi^{(d)}_{(c)}(\mathbf{r},t) = \frac{1}{4\pi} \int_0^\infty d\chi \; \chi J_0(\chi\rho) \int_0^\infty d\left(\frac{\omega}{c}\right) \frac{a_1}{\sqrt{(\omega/c)^2-\chi^2}} \, \delta\left(\left(\frac{\omega}{c}\right) \mp \sqrt{\left(\frac{\omega}{c}\right)^2 - \chi^2 - 2\beta}\right)$$
$$\times e^{-a_1((\omega/c) \pm \sqrt{(\omega/c)^2-\chi^2})/2} e^{\mp i\sqrt{(\omega/c)^2-\chi^2} z} e^{i\omega t}. \tag{2.9b}$$

In contradistinction to the Whittaker expansion, the square roots in the integrands can become imaginary if $\chi > (\omega/c)$. Thus, with a proper choice of the sign of the imaginary square root, the corresponding portions of the above integrations become superpositions of exponentially decaying evanescent components in the positive z half-space. For $\chi < (\omega/c)$, the integration yielding $\Psi^{(d)}(\mathbf{r},t)$ in Eq. (2.9) is comprised of plane waves moving in the positive z direction. Such wave components can be viewed as outgoing from an aperture situated at $z=0$. In a similar fashion, the integration leading to $\Psi^{(c)}(\mathbf{r},t)$ represents a superposition of incoming plane waves converging on the same aperture. To separate the evanescent portions of the fields represented in Eq. (2.9) from their propagating components, it is preferable to transform the integrals into the corresponding angular spectral superpositions.[20] We start by the diverging component, where the transformation

$$\chi = \kappa \sin\alpha \quad \text{and} \quad \omega/c = \kappa, \quad \text{with} \quad \sqrt{(\omega/c)^2-\chi^2} = \kappa\cos\alpha, \tag{2.10}$$

reduces $\Psi^{(d)}(\mathbf{r},t)$ to the following form:

$$\Psi^{(d)}(\mathbf{r},t) = \frac{a_1}{4\pi} \int_{D^+} d\alpha \int_0^\infty d\kappa \; \kappa \sin\alpha \; J_0(\kappa\rho\sin\alpha) e^{-\kappa a_1(1+\cos\alpha)/2}$$
$$\times \delta(\kappa - \kappa\cos\alpha - 2\beta) e^{-i\kappa(z\cos\alpha - ct)}. \tag{2.11}$$

The contour D^+, shown in Fig. 2, is chosen in the complex α plane to ensure the finiteness of the integration in the $z>0$ half-space. From the transformation relationships (2.10), it is clear that α is real as long as $\chi \leq \kappa$. Hence, the first portion of D^+ has $\alpha = \alpha_R$ and $\alpha_I = 0$, where $0 \leq \alpha_R < \pi/2$. The corresponding values of $\cos\alpha$ range from 1 to 0. The integration in Eq. (2.11) is, thus, a superposition over plane waves propagating away from the aperture into the positive z half-space. In general, the angle α is complex with $\sin(\alpha_R + i\alpha_I) = \chi/\kappa$. The second portion of the contour D^+

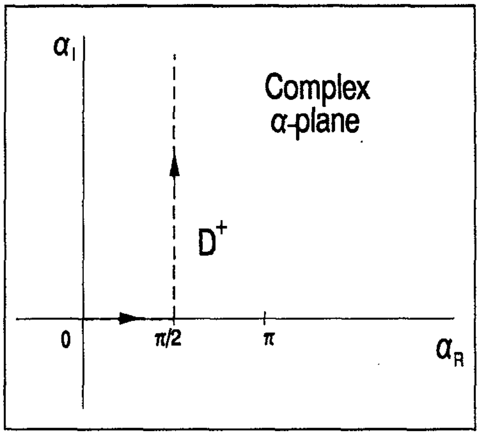

FIG. 2. The D^+ contour of integration in the complex α plane of the diverging Weyl component given in Eq. (2.11).

corresponds to $\chi \geq \kappa$ has $\alpha_R = \pi/2$, for which $\sin\alpha = \sin((\pi/2)+i\alpha_I) = \cosh\alpha_I$ and $\cos\alpha = -i\sinh\alpha_I$. The imaginary part α_I has been chosen to take only positive values between 0 and ∞, such that $\sinh\alpha_I$ stays positive. Consequently, the exponential dependence in the integrand $\exp(-i\kappa z\cos\alpha) = \exp(-\kappa z\sinh\alpha_I)$ decays to 0 as $z\to\infty$. These exponential functions do not represent propagating wave components, but inhomogeneous evanescent modes. The integration in Eq. (2.11) can now be split into two parts, one corresponding to a superposition of outgoing propagating waves and the other is an expansion in terms of the associated evanescent modes. Specifically, we have

$$\Psi^{(d)}(\mathbf{r},t) = \Psi_1^{(d)}(\mathbf{r},t) + \Psi_2^{(d)}(\mathbf{r},t), \tag{2.12a}$$

where, after carrying out the integration over κ,

$$\Psi_1^{d}(\mathbf{r},t) = \frac{\beta a_1}{2\pi}\int_0^{\pi/2} d\alpha_R \frac{\sin\alpha_R}{(1-\cos\alpha_R)^2} J_0\left(\frac{2\beta\rho\sin\alpha_R}{(1-\cos\alpha_R)}\right) e^{-\beta a_1(1+\cos\alpha_R)/(1-\cos\alpha_R)} \times e^{-i2\beta z\cos\alpha_R/(1-\cos\alpha_R)} e^{+i2\beta ct/(1-\cos\alpha_R)} \tag{2.12b}$$

and

$$\Psi_2^{(d)}(\mathbf{r},t) = \frac{i\beta a_1}{2\pi}\int_0^{\infty} d\alpha_I \frac{\cosh\alpha_I}{(1+i\sinh\alpha_I)^2} J_0\left(\frac{2\beta\rho\cosh\alpha_I}{(1+i\sinh\alpha_I)}\right) e^{-\beta a_1(1-i\sinh\alpha_I)/(1+i\sinh\alpha_I)} \times e^{-2\beta z\sinh\alpha_I/(1+i\sinh\alpha_I)} e^{+i2\beta ct/(1+i\sinh\alpha_I)}. \tag{2.12c}$$

The first component given in Eq. (2.12b) is identical to $\Psi^{(+)}(\mathbf{r},t)$ of the Whittaker representation [cf. Eq. (2.7)]. The second component given in Eq. (2.12c) is the evanescent fields associated with the diverging components. As for the Weyl converging component, we evaluate the integration in Eq. (2.9) leading to $\Psi^{(c)}(\mathbf{r},t)$. The change of variables,

$$\chi = \kappa\sin\alpha \quad \text{and} \quad \omega/c = \kappa, \quad \text{with} \quad \sqrt{(\omega/c)^2-\chi^2} = -\kappa\cos\alpha, \tag{2.13}$$

transforms $\Psi^{(c)}(\mathbf{r},t)$ in Eq. (2.9) to the following form:

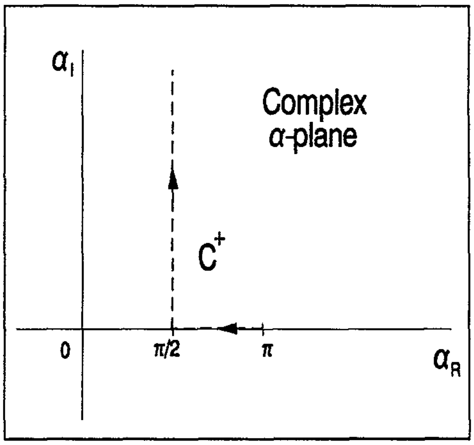

FIG. 3. The C^+ contour of integration in the complex α plane of the converging Weyl component given in Eq. (2.14).

$$\Psi^c(\mathbf{r},t)=-\frac{a_1}{4\pi}\int_{C^+}d\alpha\int_0^\infty d\kappa\ \kappa\ \sin\ \alpha\ J_0(\kappa\rho\ \sin\ \alpha)e^{-\kappa a_1(1+\cos\ \alpha)/2}$$
$$\times\delta(\kappa-\kappa\ \cos\ \alpha-2\beta)e^{-i\kappa(z\ \cos\ \alpha-ct)}. \tag{2.14}$$

Here, the contour C^+ shown in Fig. 3 corresponds to a superposition of incoming or converging waves from the $z>0$ half-space on an aperture situated at $z=0$. It should be noted that the integrand in Eq. (2.14) resembles that in Eq. (2.11), except for being negative and having the contour C^+ instead of D^+. The choice of the former contour is dictated by the need that cos α be negative when $\kappa>\chi$. So, as χ changes from 0 to κ, the real part of the angle α goes between π and $\pi/2$. The value of sin α stays positive, while cos α becomes negative, with values ranging from -1 to 0. The integration in Eq. (2.14) is, thus, a superposition over plane waves propagating toward the aperture from the positive z half-space. When $\chi>\kappa$, the angle α becomes complex with $\sin(\alpha_R+i\alpha_I)=\chi/\kappa$. For the second portion of the contour C^+, one has $\alpha_R=\pi/2$, hence, $\sin\ \alpha=\sin((\pi/2)+i\alpha_I)=\cosh\ \alpha_I$ and $\cos\ \alpha=-i\ \sinh\ \alpha_I$. The imaginary part α_I has been chosen to take only positive values between 0 and ∞, such that sinh α_I stays positive. As in the case of the Weyl diverging components, that specific choice of the contour ensures that the exponential dependence in the integrand $\exp(-i\kappa z\ \cos\ \alpha)=\exp(-\kappa z\ \sinh\ \alpha_I)$ decays to 0 as $z\rightarrow\infty$. The negative sign preceding the integration in Eq. (2.14) can be incorporated in reversing the signs of the contour integration. Using the results of the earlier calculations, we can split the Weyl converging field solution into two terms, specifically,

$$\Psi^{(c)}(\mathbf{r},t)=\Psi_1^{(c)}(\mathbf{r},t)+\Psi_2^{(c)}(\mathbf{r},t), \tag{2.15}$$

where the first term corresponds to the integration over $\alpha=\alpha_R$, taking values between π and $\pi/2$. This yields

$$\Psi_1^{(c)}(\mathbf{r},t)=\Psi^{(-)}(\mathbf{r},t). \tag{2.16}$$

The integration over the rest of the contour can be calculated, as in the case of the Weyl diverging component, to give

$$\Psi_2^{(c)}(\mathbf{r},t)=-\Psi_2^{(d)}(\mathbf{r},t). \tag{2.17}$$

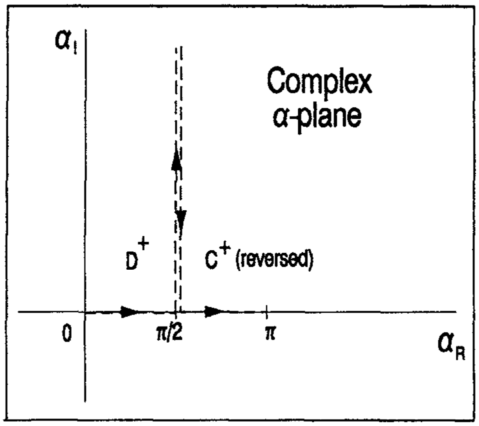

FIG. 4. The D^+ contour, together with the reversed C^+ contour, in the complex α plane.

As such, the Weyl diverging and converging components add up resulting in the source-free solution,

$$\Psi_{sf}(\mathbf{r},t)=\Psi^{(d)}(\mathbf{r},t)+\Psi^{(c)}(\mathbf{r},t)=\Psi^{(+)}(\mathbf{r},t)+\Psi^{(-)}(\mathbf{r},t). \tag{2.18}$$

Thus, the source-free FWM modes do not have any evanescent field components associated with them. The total evanescent field is given specifically by the summation

$$\Psi_2(\mathbf{r},t)=\Psi_2^{(d)}(\mathbf{r},t)+\Psi_2^{(c)}(\mathbf{r},t)=0. \tag{2.19}$$

This means that the total evanescent field associated with the source-free FWM is equal to zero. This point can be better understood if we refer back to the contours D^+ and C^+ in the complex α plane. Such contours have been used to evaluate $\Psi^{(d)}(\mathbf{r},t)$ and $\Psi^{(c)}(\mathbf{r},t)$ given in Eqs. (2.11) and (2.14). The integrands in the aforementioned expressions differ only by a negative sign. The difference in the sign can be accommodated into the reversal of the sense of the contour integration over C^+. With the two contours plotted together, it can be seen from Fig. 4 that the contour integrations contributing to the evanescent components cancel each other. The integrations over α_I, thus, subtract from each other, and we are left with the integrations over α_R ranging from $0\rightarrow\pi$. This leads to the Whittaker superposition of forward and backward propagating components. Hence, the diverging field of the Weyl expansion cannot be interpreted as the field generated by an aperture situated at $z=0$ and propagating into the positive z half-space. This is the case, because the integration over the δ function in the spectrum [cf. Eq. (2.9)] transforms the evanescent fields into propagating ones, thus depriving the Weyl representation of its physical meaning and mathematical form. Similar results has been obtained in Ref. 24 in connection to spherical waves and their representations in the Whittaker and the Weyl expansions. To generate the FWM in a completely causal sense, we have to consider an initial FWM excitation of an aperture, as will be done in Sec. IV. The resulting causal field, propagating in the $z>0$ half-space, will be a very good approximation of the source-free FWM, under the condition that $\beta a_1 \ll 1$. Nevertheless, we need to study first the causality of the FWM solution and to derive an estimate for the relative strength of its forward versus its backward traveling components for the various limits of the parameter βa_1.

III. THE CAUSALITY OF THE FOCUS WAVE MODES

The causality of the FWM has been dealt with in earlier works,[17,18] where it has been shown that under the specific condition of $\beta a_1 \gg 1$, the acausal components are dominant. Such a conditioned conclusion could be reached by referring directly to Fig. 1(b), where it is shown that almost all the spectral content of the FWM lies in the acausal portion of the spectrum. It has been concluded,[18] however, that the predominance of the acausal components (under the $\beta a_1 \gg 1$ condition) over the causal ones poses serious problems for the realization of any physical FWM fields. In the aforementioned work, the other limiting case of $\beta a_1 \ll 1$ has not been considered. Furthermore, since a large number of the LW solutions are superpositions of the source-free FWM, they were deemed[18] unlaunchable, because of their dilemma with causality. We believe, however, that such a conclusion has been reached for a specific case, and it is incorrect to pass the same judgement for all other LW solutions. In this section, we show that the source-free FWM itself is predominantly causal when $\beta a_1 \ll 1$. Other LW solutions become essentially causal under a similar choice of parameters. In the following section we use the same results to show that a completely causal FWM excitation of an aperture will produce a field that approximates to a great extent the source-free FWM. This alludes to fact that the behavior of the experimentally generated LW[15,16] agrees very well with the theoretical predictions.

The analysis of the angular spectral content reflected in Figs. 1(a) and 1(b), shows that the causal (acausal) components are dominant when $\beta a_1 \ll 1$ ($\beta a_1 \gg 1$). The same conclusion has been reached[3,14] earlier by comparing the Fourier spectrum content of the forward and the backward traveling components of the source-free FWM. However, a rigorous quantitative estimate for the relative strength of such components has not been derived. In this section, we derive an asymptotic estimate of the relative strength of the forward versus the backward components of the source-free FWM, under the two extreme conditions of $\beta a_1 \gg 1$ and $\beta a_1 \ll 1$. The central portion of the FWM pulse for which $z - ct < a_1$, and $\rho < \sqrt{a_1/\beta}$ is the part of the field that has the highest intensity values. Hence, we shall concentrate our attention on the $z = ct$ central portion of the pulse.

Starting with the $\beta a_1 \gg 1$ limit, which has been considered by Heyman and Felsen, we can rewrite the forward propagating component in Eq. (2.8) as

$$\Psi^{(+)}(\rho, z=ct) = \frac{\beta a_1}{2\pi} \int_1^\infty d\lambda\ \lambda J_0(2\beta\rho\lambda) e^{-\beta a_1 \lambda^2} e^{i2\beta ct}. \tag{3.1}$$

For $\beta a_1 \gg 1$, the leading-order term in an asymptotic expansion of the Laplace-type integration given in Eq. (3.1) can be evaluated using the procedure, resulting in Eq. (6.4.19a) in Ref. 25. For the integration (3.1), the Laplace variable $\Phi(\lambda) = -\lambda^2$ is maximum at the end point $\lambda = 1$. Hence, it follows that[25]

$$\Psi^{(+)}(\rho, z=ct) \sim \frac{e^{-\beta a_1}}{4\pi} J_0(2\beta\rho) e^{i2\beta ct}. \tag{3.2}$$

To determine the relative contribution of the $\Psi^{(-)}$ component, we start with

$$\Psi^{(-)}(\rho, z=ct) = \frac{\beta a_1}{2\pi} \int_0^1 d\lambda\ \lambda J_0(2\beta\rho\lambda) e^{-\beta a_1 \lambda^2} e^{i2\beta ct}. \tag{3.3}$$

The new variable $s = \lambda^2$ transforms Eq. (3.3) into

$$\Psi^{(-)}(\rho, z=ct) = \frac{\beta a_1}{4\pi} e^{i2\beta ct} \int_0^1 ds\ e^{-s\beta a_1} J_0(2\beta\rho\sqrt{s}). \tag{3.4}$$

Most of the contribution to this Laplace-type integral comes from around $s = 0$. Thus, it is possible to asymptotically expand the upper limit of the integration to ∞, and using the series expansion of the Bessel function, we get

$$\Psi^{(-)}(\rho,z=ct)\sim\frac{\beta a_1}{4\pi}\,e^{i2\beta ct}\int_0^\infty ds\ e^{-s\beta a_1}\sum_{k=0}^{\infty}\frac{(-1)^k(\beta\rho)^{2k}}{k!\Gamma(k+1)}\,s^k. \tag{3.5}$$

Integrating over s reduces Eq. (3.5) to

$$\Psi^{(-)}(\rho,z=ct)\sim\frac{\beta a_1}{4\pi}\,e^{i2\beta ct}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{\beta\rho^2}{a_1}\right)^k. \tag{3.6}$$

The series definition of the exponential function gives the following expression for the backward field component of the source-free FWM, namely,

$$\Psi^{(-)}(\rho,z=ct)\sim\frac{1}{4\pi}\,e^{i2\beta ct}e^{-\beta\rho^2/a_1}. \tag{3.7}$$

From Eqs. (3.2) and (3.7), we obtain the relative magnitude of the positive to the negative going components, viz.,

$$\frac{\Psi^{(+)}(\rho,z=ct)}{\Psi^{(-)}(\rho,z=ct)}\sim\frac{e^{-\beta a_1}}{e^{-\beta\rho^2/a_1}}\,J_0(2\beta\rho). \tag{3.8}$$

Around $\rho=0$, we have $J_0(2\beta\rho)\sim O(1)$ and $e^{-\beta\rho^2/a_1}\sim O(1)$, hence

$$\frac{\Psi^{(+)}(\rho,z=ct)}{\Psi^{(-)}(\rho,z=ct)}\sim e^{-\beta a_1} \tag{3.9}$$

This shows that, under the condition $\beta a_1 \gg 1$, the causal outgoing component is exponentially small. However, such a component can become dominant for all values of $J_0(2\beta\rho)\exp(-\beta a_1)>\exp(-\beta\rho 2/a_1)$. This occurs far away from the center of the pulse when $\rho > a_1 \gg \sqrt{a_1/\beta}$. The same conclusions have been reached in Ref. 18, in which the other possibility of $\beta a_1 \ll 1$ has not been considered. Next, we demonstrate that in the latter case, the causal outgoing field is dominant.

In the other limit, with $\beta a_1 \ll 1$, one can follow a similar procedure to determine the relative magnitude of the positive and negative going components. Starting with Eq. (3.1), we rewrite it as follows:

$$\Psi^{(+)}(\rho,z=ct)=\frac{\beta a_1}{2\pi}\int_0^\infty d\lambda\ \lambda J_0(2\beta\rho\lambda)e^{-\beta a_1\lambda^2}e^{i2\beta ct}-\Psi^{(-)}(\rho,z=ct)$$

$$=\frac{1}{4\pi}\,e^{-\beta\rho^2/a_1}e^{i2\beta ct}-\Psi^{(-)}(\rho,z=ct), \tag{3.10}$$

where $\Psi^{(-)}(\rho,z=ct)$ is given in Eq. (3.3). An estimate for the acausal negative going component of the FWM can be calculated by imposing the condition $\beta a_1 \ll 1$, for which $\exp(-\beta a_1\lambda^2)\simeq 1$, in the range $0\leqslant\lambda\leqslant 1$. Specifically, the integration in Eq. (3.3) may be approximated by

$$\Psi^{(-)}(\rho,z=ct)=\frac{\beta a_1}{2\pi}\int_0^1 d\lambda\ \lambda J_0(2\beta\rho\lambda)e^{i2\beta ct}. \tag{3.11}$$

Integrating over λ and combining Eqs. (3.10) and (3.11), we obtain

$$\frac{\Psi^{(+)}(\rho,z=ct)}{\Psi^{(-)}(\rho,z=ct)} \sim \left\{ \frac{\rho}{a_1} \frac{e^{-\beta\rho^2/a_1}}{J_1(2\beta\rho)} - 1 \right\}. \tag{3.12}$$

For the central portion of the FWM pulse, we have $\rho \leqslant 2\sqrt{a_1/\beta}$. Inside such a focused waist, the focused FWM field has its highest intensity and the Bessel function $J_1(2\beta\rho) \leqslant J_1(4\sqrt{\beta a_1})$. The condition $\beta a_1 \ll 1$, thus, yields the small argument asymptotic estimate $J_1(2\beta\rho) \sim \beta\rho$. Consequently, Eq. (3.12) may be rewritten as

$$\frac{\Psi^{(+)}(\rho,z=ct)}{\Psi^{(-)}(\rho,z=ct)} \sim \left\{ \frac{1}{\beta a_1} e^{-\beta\rho^2/a_1} - 1 \right\}. \tag{3.13}$$

If βa_1 is chosen such that $\beta a_1 \ll e^{-4} < 1$, then the second term on the right-hand side may be dropped for $\rho \leqslant 2\sqrt{a_1/\beta}$. Thus, the forward propagating component is algebraically larger than the backward traveling one. This shows that the forward going component is predominant within the central portion of the FWM field. Such a behavior is crucial in determining the shape of the causal FWM pulse generated by an aperture (cf. Sec. IV). In the next section, it will be demonstrated that the radiated FWM pulse closely resembles the source-free field; especially in the high-intensity portion surrounding its centroid. Subsequently, the form of the two pulses differs only in the tails of their fields, characterized by having much smaller amplitudes. Furthermore, for the narrower waist $\rho \leqslant 2\sqrt{a_1/\beta}$, the ratio Eq. (3.13) acquires the following form:

$$\frac{\Psi^{(+)}(\rho,z=ct)}{\Psi^{(-)}(\rho,z=ct)} \sim \frac{1}{\beta a_1}. \tag{3.14}$$

Thus, it is established that, under the condition $\beta a_1 \ll 1$, the causal positive going component is dominant around the centroid of the pulse. The relative magnitude of the acausal component is algebraically smaller than the causal one. In contradistinction, Eq. (3.9), which is valid for $\beta a_1 \gg 1$, indicates that the causal component is exponentially small relative to the acausal one. Such a difference in behavior between the two extreme cases is a consequence of the sharp rising edges of the spectra shown in Figs. 1(a) and 1(b), in contrast with their extended tails that vanish at $\alpha = \pi$. For example, when $\beta a_1 = 10$, there are *almost* no spectral components in the causal domain, while for $\beta a_1 = 0.1$ there is still a significant portion of the spectrum in the acausal range. However, for $\beta a_1 = 0.01$ the acausal components are reduced significantly. One should also note that the latter value of βa_1 barely satisfies the stricter condition discussed in the paragraph following Eq. (3.13). Obviously, the condition $\beta a_1 \ll e^{-4}$ enforces the predominance of the forward going causal components of the FWM field. Having established that for $\beta a_1 \ll 1$ the acausal components around the center of the FWM pulse are insignificant, we proceed further to demonstrate the possibility of causally radiating good approximations to such a solution from a flat aperture.

IV. THE CAUSAL EXCITATION OF THE FWM APERTURE

In a previous work by the authors,[14] it has been shown that a Huygens construction of a Bessel beam[11] generated from an infinite aperture cancels out all acausal incoming components, and the Bessel beam is propagated invariantly away from the aperture. The FWM can be considered as a superposition of Bessel beams.[3] Thus, the same approach could be applied to the FWM to show that no acausal incoming fields are generated. In fact, it will be demonstrated that a good approximation to the source-free FWM can be generated from an aperture that shrinks from an infinite size to an effective minimum radius of $2\sqrt{a_1/\beta}$, which, henceforth, expands once more to infinity. Such a time varying aperture requires an infinite time of illumination. This is the main reason for the need of an infinite amount of energy to generate such a field. Nevertheless, the generated FWM field does not need infinite power to illuminate its aperture.

We start by defining the initial FWM field on an aperture situated at the $z=0$. The field is the real part of the azimuthally symmetric complex FWM pulse; specifically,

$$\Psi_i(\rho,t)=\frac{a_1}{4\pi(a_1-ict)}\, e^{-\beta\rho^2/(a_1-ict)}e^{i\beta ct}. \tag{4.1}$$

Because of the first exponential term on the right-hand side, the field exists mainly inside the radius,

$$R(t)=2\,\frac{|ct|}{\sqrt{\beta a_1}}, \quad \text{for } ct\gg a_1. \tag{4.2}$$

It is clear that such a time varying radius becomes infinite when $ct=\pm\infty$. On the other hand, the radius $R(t)\sim 2\sqrt{a_1/\beta}$ when t approaches the value of zero. Furthermore, it can be deduced from Eq. (4.1) that the field amplitude varies as $1/ct$ as $ct\rightarrow\infty$. Subsequently, the intensity of the illuminating field is reduced as $(1/ct)^2$ and becomes equal to zero at $t=\pm\infty$. This means that as $ct\rightarrow\infty$, the power of the field illuminating the aperture (=the intensity×the area of the aperture) remains constant and finite. This behavior should be compared to the illumination of infinite apertures by plane waves or Bessel beams. In these two cases, we need infinite power to illuminate the corresponding apertures. From Eq. (4.2), it can be seen that the parameter βa_1 controls the speed of the shrinking and expanding of the aperture, where

$$v_{\text{ap}}=\frac{2c}{\sqrt{\beta a_1}}, \quad \text{for } ct\gg a_1. \tag{4.3}$$

For the case of $\beta a_1<1$, the FWM aperture expands effectively at a speed $v_{\text{ap}}>c$. This is achieved by composing the aperture from separately excitable elements. This has been the case for the arrays used in experiments performed to establish the launchability of the LW solutions.[15,16]

The Fourier spectrum of the illumination of the FWM aperture is calculated by finding the Fourier transform of Eq. (4.1), viz.,

$$\Phi_{\text{ap}}(\chi,\omega)=\int_{-\infty}^{\infty}dt\int_0^{\infty}d\rho\;\rho J_0(\chi\rho)e^{-i\omega t}\,\frac{a_1}{4\pi(a_1-ict)}\,e^{-\beta\rho^2/(a_1-ict)}e^{i\beta ct}. \tag{4.4}$$

The integrations over t and ρ, when carried out, yield

$$\Phi_{\text{ap}}(\chi,\omega)=\frac{a_1}{4\beta}\,\delta(\omega-[(\chi^2/4\beta)+\beta]c)e^{-(\chi^2/4\beta)a_1}. \tag{4.5}$$

Notice that the spectrum given in Eq. (4.5) differs from the corresponding spectrum of the diverging and converging Weyl spectra used in Eqs. (2.9). The roots of the δ functions in both cases are the same. Nevertheless, each spectrum corresponds to a distinct physical situation. One is associated with a source-free pulse and the other with an aperture generated field.

To define the normal derivative of the field illuminating the aperture, we can use the source-free Whittaker representation given in Eq. (2.6). The differentiation of the field with respect to z is evaluated, and then we set $z=0$. According to the analysis presented in Appendix B of Ref. 14, the negative going (acausal) component $\Psi^{(-)}$ is filtered out, and *only* the positive going (causal) component $\Psi^{(+)}$ is propagated in the positive z direction. If we use the full wave representation (2.6) of the FWM to define the initial field and its normal derivative, it is straightforward to show that the generated field $\Psi(\mathbf{r},t)$ is free from any evanescent modes. Such a conclusion follows from the fact that the initial field illuminating the aperture does not have any evanescent fields associated with it [cf. Sec. II], hence the radiated field $\Psi(\mathbf{r},t)$ is also free from such field components. The same conclusion might not be obvious if the normal derivative of the field illuminating the aperture is defined in a slightly different manner. We choose to define the initial field on the aperture [cf. Eq. (4.1)] as a superposition of Bessel beams at $z=0$; specifically,

$$\Psi_i(\rho,t)=\left(\frac{1}{2\pi}\int_0^\infty d\chi\,\chi J_0(\chi\rho)\int_0^\infty d\omega\,\Phi_{\rm ap}(\chi,\omega)e^{-i\sqrt{(\omega/c)^2-\chi^2}z}e^{i\omega t}\right)_{z=0}. \tag{4.6}$$

The normal derivative on the aperture is now defined as the differentiation of the above expression with respect to z before setting $z=0$. This representation differs from (2.6) by restricting $\sqrt{(\omega/c)^2-\chi^2}$ to positive values *only*. Such a condition ensures the forward illumination of the aperture, i.e., all the spectral components are propagated away from the aperture in a causal manner. In contradistinction, if the representation (2.6) is used, the $\Psi^{(-)}$ Whittaker component is never launched out of the aperture into the positive z half-space. Under the condition, $\beta a_1 \ll 1$, the radiated field is approximately the same in both excitation schemes, because the magnitude of the causally generated field components equivalent to the Whittaker $\Psi^{(-)}$ field is negligible in comparison to $\Psi^{(+)}$. This fact will be made clear from the details of our analysis. Notice, also, that ω is restricted to positive values if β is positive. This follows from the roots of the δ function in the spectrum given in Eq. (4.5).

To calculate the outgoing field propagating into the $z>0$ half-space, Huygens construction is applied to the initial excitation of the aperture. Accordingly, the field at a point $\mathbf{R}$ and time t inside a wave front surface having a zero field outside such a surface is given by the integration over the area of the infinite aperture,

$$\Psi_{\rm ap}(\rho,z,t)=\frac{1}{4\pi}\int_0^{2\pi}d\phi'\int_0^\infty d\rho'\,\frac{\rho'}{R}\bigg(-\partial_{z'}\Psi(\rho',z'=0,t')+\frac{z}{R^2}\Psi(\rho',z'=0,t')$$
$$+\frac{z}{cR}\partial_{t'}\Psi(\rho',z'=0,t')\bigg)_{t'=t-R/c}. \tag{4.7}$$

The primed coordinates refer to source points on the aperture, while the unprimed ones refer to the observation points in the $z>0$ half-space. Substituting for the spectral expansion given in Eq. (4.6), we get

$$\Psi_{\rm ap}(\rho,z,t)=\frac{1}{4\pi}\int_0^{2\pi}d\phi'\int_0^\infty d\rho'\,\frac{\rho'}{2\pi}\int_0^\infty d\chi\,\chi J_0(\chi\rho')\int_0^\infty d\omega\,\Phi_{\rm ap}(\chi,\omega)e^{i\omega t}$$
$$\times\left\{i\sqrt{\left(\frac{\omega}{c}\right)^2-\chi^2}\,\frac{e^{-i\omega R/c}}{R}-\partial_z\left(\frac{e^{-i\omega R/c}}{R}\right)\right\}. \tag{4.8}$$

Here $R=\sqrt{\rho'^2+\rho^2-2\rho'\rho\cos\phi'+z^2}$. In Appendix A, we provide the details of the derivation of a more tractable form of the radiated field [cf. Eq. (A5)]. Since the square root $\sqrt{(\omega/c)^2-\chi^2}$ is always positive, it forces the bracketed term in Eq. (4.8) to pick up only outgoing waves, and to cancel out any waves converging on the aperture. Such a condition is crucial for the evaluation of the integration given in Eq. (A5); specifically, one should note that $\sqrt{(\omega/c)^2-\chi^2}=(\chi^2/4\beta)-\beta$ for $\chi>\beta$, while $\sqrt{(\omega/c)^2-\chi^2}=-(\chi^2/4\beta)+\beta$ for $\chi<\beta$. The integration over the ω variable in Eq. (A5) leads to

$$\Psi_{\rm ap}(\mathbf{r},t)=\frac{\beta a_1}{2\pi}\int_1^\infty d\lambda\ \lambda J_0(2\beta\rho\lambda)e^{-\beta[a_1+i(z-ct)]\lambda^2}e^{i\beta(z+ct)}$$

$$+\frac{\beta a_1}{2\pi}\int_0^1 d\lambda\ \lambda J_0(2\beta\rho\lambda)e^{-\beta[a_1-i(z+ct)]\lambda^2}e^{-i\beta(z-ct)}. \tag{4.9}$$

Since the spectral content of the two parts of the solution is the same as the source-free FWM, then one should deduce that for $\beta a_1 \ll 1$, the first term is dominant. In such a limit, Eq. (4.9) can be rewritten as

$$\Psi_{\rm ap}(\mathbf{r},t)=\Psi_{\rm sf}(\mathbf{r},t)+\Psi_\epsilon(\mathbf{r},t). \tag{4.10}$$

For $\beta a_1 \ll 1$, the second term on the right-hand side is a small term of order $O(\beta a_1/\pi)$ relative to the first, and is given explicitly by

$$\Psi_\epsilon(\mathbf{r},t)=i\,\frac{\beta a_1}{\pi}\int_0^1 d\lambda\ \lambda J_0(2\beta\rho\lambda)e^{-\beta a_1\lambda^2}\sin[(\lambda^2-1)\beta z]e^{+i(\lambda^2+1)\beta ct}. \tag{4.11}$$

Thus, the causal field excited from an aperture situated at $z=0$ resembles [within an error of order $(\beta a_1/\pi)$] the field of the source-free FWM.

Notice that the superposition given in Eq. (4.9) does not contain any evanescent field components, and is constituted solely from propagating ones. As discussed earlier, the use of the full wave representation (2.6) to define the initial field and its normal derivative leads to an aperture radiated field $\Psi(\mathbf{r},t)$ free from any evanescent modes. Such a conclusion follows from the fact that the aperture illumination field does not have any evanescent fields associated with it [cf. Sec. II], hence the radiated field $\Psi(\mathbf{r},t)$ is also free from such field components. The same conclusion might not be obvious if the normal derivative of the field illuminating the aperture is defined alternatively by Eq. (4.6). In Appendix B we give a detailed proof of the annulment of the evanescent fields components for the excitation field given in Eq. (4.6). Even though the result of Appendix B is same as in the case of the source-free FWM, nevertheless, the underlying physics is completely different. In the case of the source-free solution, the evanescent fields associated with the Weyl converging components cancel identically those associated with the Weyl diverging components. For the FWM aperture it can be shown [cf. Eqs. (B6–B8)] that the annulment of the evanescent fields is due to the proper choice of the poles of k_z, thus ensuring the causality of the generated fields. Nevertheless, the nonexistence of the evanescent fields can be perceived as a direct result of the infinite size of the aperture. The infinite time of excitation produces the δ function in the spectrum (4.5). Such a δ function enforces the condition $(\omega/c)>\chi$, while the other situation $(\omega/c)<\chi$ becomes superfluous and leads to zero evanescent fields. Thus, we expect that a FWM aperture expanding for a finite period of time, $-T<t<T$, should generate evanescent fields. The finiteness of the period of excitation removes the δ function from the spectrum. If $\delta(\omega-[(\chi^2/4\beta)+\beta]c)$ in Eq. (4.5) is replaced by a regular function, then k does not become complex, and consequently the poles of k_z in Eq. (B5) take only imaginary values. The contour integration over D_{01}^+ or D_{02}^+ picks up one of the two poles of k_z given in Eq. (4.23). The evanescent fields are, thus, real and are not transformed back into propagating components by the action of the δ function.

To qualify our claims, we multiply the initial excitation $\Psi_i(\rho,t)$ given in Eq. (4.1) by the Gaussian time window $\exp(-t^2/4T^2)$, hence limiting the time of expansion of the aperture. In this case, the spectrum of the field illuminating the aperture becomes

$$\Phi_{\rm ap}(\chi,\omega)=\frac{a_1}{4\beta}\,\hat{\delta}\left(\omega-\left[\left(\frac{\chi^2}{4\beta}\right)+\beta\right]c;T\right)e^{-(\chi^2/4\beta)a_1}, \tag{4.12}$$

where

$$\hat{\delta}\left(\omega-\left[\left(\frac{\chi^2}{4\beta}\right)+\beta\right]c;T\right)=\frac{T}{\sqrt{\pi}}\,e^{-T^2(\omega-[(\chi^2/4\beta)+\beta]c)^2}. \tag{4.13}$$

In the limit $T\to\infty$, the expression on the right-hand side of Eq. (4.13) reduces to the Dirac δ function of Eq. (4.5). For finite T, the integrand in Eq. (B1) contains the $\hat{\delta}$ function instead of the Dirac δ function. The evanescent field associated with the finite-time FWM aperture is given explicitly as

$$\Psi_0(\mathbf{r},t)=-\frac{ia_1}{8\pi\beta}\int_0^\infty d\kappa\int_0^\infty d\alpha_I\,\kappa^2\sinh\alpha_I\cosh\alpha_I e^{-(\kappa^2/4\beta)a_1\cosh^2\alpha_I}J_0(\kappa\rho\cosh\alpha_I)$$
$$\times\frac{T}{\sqrt{\pi}}\,e^{-T^2(\kappa-(\kappa^2/4\beta)\cosh^2\alpha_I-\beta)^2}e^{i\kappa ct}e^{-\kappa\sinh\alpha_I z}. \tag{4.14}$$

There is no way for such integration to cancel out in the same delicate manner that leads to Eq. (B8). Furthermore, the $\exp(-\kappa\sinh\alpha_I z)$ term represents real evanescent modes. Thus, a finite-time expanding FWM aperture has evanescent field components associated with it.

V. CONCLUSIONS

In this work, we have studied the angular spectral content of the FWM solution to the scalar wave equation. Such an approach has proven to be a suitable vehicle to address issues, like the causality of such solutions, the possibility of generating them from an aperture, and the nature of the evanescent fields associated with them. Two distinct cases have been investigated in detail, the source-free FWM solution and an aperture excitation by an initial FWM field. Even though the two situations are quite different, it has been demonstrated that for an appropriate choice of parameters (viz. $\beta a_1\ll 1$) there is a great resemblance between the two fields.

The source-free FWM solution has been studied using both the Whittaker and the Weyl representations. The former uses a superposition of outgoing and incoming plane waves, while the latter is composed of plane waves diverging and converging from an aperture, together with the associated evanescent components. It has been demonstrated that the diverging and converging Weyl components add up in such a way that the evanescent modes cancel out identically. Thus, we are left with the propagating components that are equivalent to the Whittaker outgoing and incoming fields. Consequently, there are no evanescent fields associated with the source-free FWM solution, the same conclusion has been reached by other authors.[17,19] On the other hand, any finite-energy superposition over the source-free FWM is free of any evanescent fields. Such a superposition sums up the evanescent components associated with the Weyl diverging and converging fields; those cancel out identically, as shown in Sec. II.

An asymptotic estimation of the relative strength of the Whittaker forward and backward traveling components has been carried out. It has been established that in the limit $\beta a_1\ll 1$, the forward propagating (causal) components are dominant over the backward traveling (acausal) ones. In fact, the acausal field around the center of the pulse, where the field is strongest, is algebraically small, in comparison to the causal one. Such a limit for βa_1 has not been considered in an earlier study of the causality of the source-free FWM solution.[18] In confirmation to the results of Heyman,[18] we have shown that the Whittaker forward propagating components are exponentially small for the other extreme; namely, for the choice of parameters giving $\beta a_1\gg 1$. Such conclusions have been made more transparent by considering the angular spectral content as a function of the angle α. Unlike the Fourier synthesis, the ranges of α contributing to the forward and backward traveling fields are finite and equal. Consequently, the choice of parameters producing fields having stronger spectral components in the forward range results in source-free solutions that are primarily moving in the positive z direction. As such, it has been shown in Fig. 1(a) that the source-free FWM solution has *almost* all its spectral components in the forward (causal) range, when $\beta a_1<0.01$.

In general, using the procedure presented in this paper, we can always identify the Weyl diverging and evanescent field components of any finite-energy localized wave solution. Such field components are the ones that could be realized by physical apertures. Nevertheless, the FWM is an exception because of its highly singular spectrum containing a Dirac δ function. Such a function transforms the diverging Weyl inhomogeneous evanescent fields into propagating ones. Thus, using the Weyl diverging components *alone* to represent the field generated from a physical aperture is not possible.

As an alternative, we have considered the case of an aperture illuminated by an initial FWM field. Such an aperture has a size that varies with time, where its radius shrinks from infinity at $t=-\infty$ to $2\sqrt{a_1/\beta}$ at $t=0$, and then expands once more to an infinite size as $t\rightarrow\infty$. Such an aperture does not need infinite power to illuminate it, instead it needs infinite energy *only* because it is excited for an infinitely long time. Such an infinite time of illumination is the reason for having a Dirac δ function in the spectrum. The causal field generated by such an aperture has been calculated using Huygens construction. It has been shown that such a field does not contain any acausal components. Furthermore, we have demonstrated that the FWM aperture field is a very good approximation of the source-free FWM solution when $\beta a_1\ll 1$. Thus, one can conclude that as far as causality is concerned, there are no "grave" problems with the generation of the FWM field, or of a very good approximation of it.

The evanescent fields associated with the FWM aperture has been found to be equal to zero. This follows from the condition of the forward initial illumination of the aperture. It can also be considered as a consequence of the infinite size of the FWM aperture. If the period of expansion is limited to the finite range $-T<t<T$, then the Dirac δ function in the spectrum becomes a regular function, the evanescent field components become real, and there is no way for them to cancel out identically. The finite-time (or finite-energy) FWM aperture is a physically realizable source, which is very efficient in generating very narrow beams from much larger extended apertures. A detailed study of such a system is deferred for future work. However, the idea of using a finite-time dynamic Gaussian aperture to generate fields that approximate the source-free FWM can be considered as a physically realizable scheme to launch LW pulses. Such an approach should be compared to other suggested methods to launch approximations to the FWM field, like generating acoustic pulses from sources moving close to the speed of sound.[27] In another attempt, it has been suggested that an approximate FWM pulse can be generated using infinite line sources,[28] such a scheme has been based on a Green's function approach. In contradistinction, the method described in this paper depends on the specification of the initial conditions of the field, illuminating a dynamic Gaussian aperture.

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APPENDIX A: THE CAUSAL GENERATION OF THE FWM PULSE

In this appendix, we demonstrate, from first principles, that the Huygens construction (4.7) leads to the generation of a fully causal FWM pulse. We start with the identity

$$\frac{e^{-i\omega R/c}}{R}=\frac{1}{\pi}\int_0^\infty d\lambda\int_{-\infty}^\infty dk_z\,\lambda J_0(\lambda\rho^*)\,\frac{e^{-ik_z z}}{k_z^2-[(\omega/c)^2-\lambda^2]},\tag{A1}$$

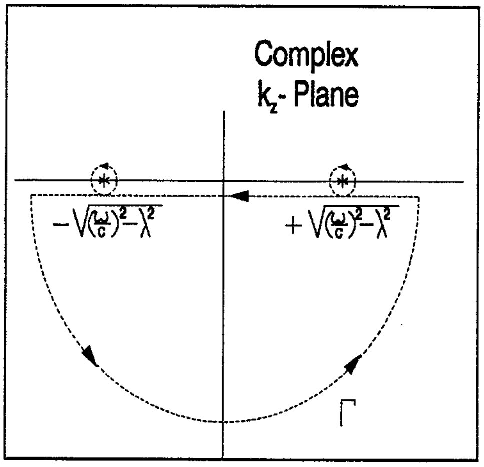

FIG. 5. The contour of integration Γ in the complex k_z plane used to evaluate the integration in Eq. (A1).

to evaluate the bracketed term in Eq. (4.8). Here $\rho^* = \sqrt{\rho'^2+\rho^2-2\rho'\rho\cos\phi'}$, and we use the contour of integration in the complex k_z plane shown in Fig. 5. For $(\omega/c)>\lambda$, the contour of integration is closed in the lower half-plane to ensure the integrability of Eq. (A1) for $z>0$. For $(\omega/c)<\lambda$, the analysis becomes more complicated, because the δ function in the spectrum given in Eq. (4.5) forces ω to become complex. Such a situation will be dealt with in Appendix B. Upon carrying out the contour integration over the k_z variable and evaluating the partial derivative with respect to z, we get

$$\partial_z\left(\frac{e^{-i\omega R/c}}{R}\right)=-\int_0^\infty d\lambda\ \lambda J_0(\lambda\rho^*)e^{-i\sqrt{(\omega/c)^2-\lambda^2}z}-\int_0^\infty d\lambda\ \lambda J_0(\lambda\rho^*)e^{+i\sqrt{(\omega/c)^2-\lambda^2}z}. \quad \text{(A2)}$$

From the *a priori* knowledge of the appearance of $\delta(\lambda-\chi)$ [cf. Eq. (A4) below], the bracketed term in Eq. (4.8) becomes

$$\left\{i\sqrt{\left(\frac{\omega}{c}\right)^2-\chi^2}\,\frac{e^{-i\omega R/c}}{R}-\partial_z\left(\frac{e^{-i\omega R/c}}{R}\right)\right\}=2\int_0^\infty d\lambda\ \lambda J_0(\lambda\rho^*)e^{-i\sqrt{(\omega/c)^2-\lambda^2}z}. \quad \text{(A3)}$$

If $\sqrt{(\omega/c)^2-\chi^2}$ is allowed to take negative values, the bracketed quantity on the left-hand side is equal to zero [cf. Appendix B in Ref. 14]. Thus, any acausal components converging on the aperture are filtered out. For the specific choice of Eq. (4.6), all the spectral components are launched out of the aperture. This is a direct consequence of the forward illumination of the FWM aperture; viz., restricting $\sqrt{(\omega/c)^2-\chi^2}$ in Eq. (4.8) to positive values only. Hence, one expects that in case ω becomes complex, only poles of k_z having *positive* real and *negative* imaginary parts contribute to the integration (4.8). This is an issue that will be addressed in Appendix B, where we consider the case of $(\omega/c)<\chi$ usually associated with the evanescent fields.

Following the same procedure as in Ref. 14, we can use the addition theorem of the Bessel function,[26] together with Eq. (A3), to rewrite Eq. (4.8) as follows:

$$\Psi_{\text{ap}}(\rho,z,t)=\frac{1}{2\pi}\int_0^\infty d\rho'\ \rho'\int_0^\infty d\chi\ \chi J_0(\chi\rho')\int_0^\infty d\omega\ \Phi_{\text{ap}}(\chi,\omega)e^{i\omega t}$$
$$\times\int_0^\infty d\lambda\ \lambda J_0(\lambda\rho)J_0(\lambda\rho')e^{-i\sqrt{(\omega/c)^2-\lambda^2}z}. \quad \text{(A4)}$$

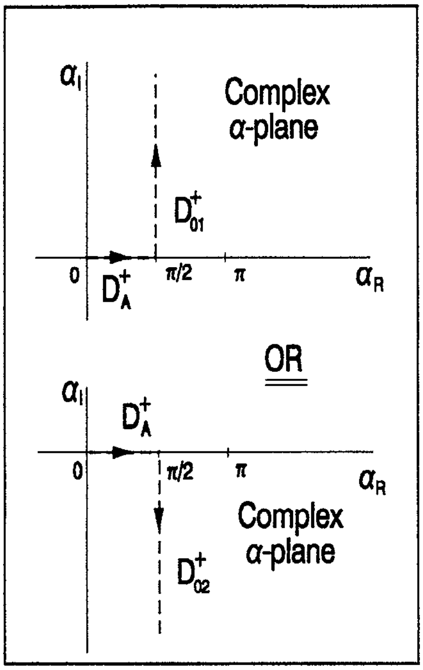

FIG. 6. The D^+_{01} and the D^+_{02} contours of integration in the complex α plane used to evaluate the integration in Eq. (B4).

The integration over ρ' can be evaluated using the orthogonality of the Bessel function, yielding a $\delta(\lambda-\chi)$ term. Thus, upon integrating over ρ' and λ, Eq. (A4) is reduced to

$$\Psi_{\rm ap}(\rho,z,t)=\frac{a_1}{8\pi\beta}\int_0^\infty d\chi\ \chi J_0(\chi\rho)\int_0^\infty d\omega\ \delta\left(\omega-\left[\left(\frac{\chi^2}{4\beta}\right)+\beta\right]c\right)e^{-(\chi^2/4\beta)a_1}e^{i\omega t}e^{-i\sqrt{(\omega/c)^2-\chi^2}z}. \tag{A5}$$

Although Eq. (A5) resembles the Weyl expansion of the source-free FWM given in Eq. (2.9), nevertheless, each expression leads to a significantly distinct physical situation. The superposition in Eq. (2.9) yields the source-free solution, while the result given in Eq. (A5) is a causal FWM field radiated from an aperture situated at $z=0$ into the $z>0$ half-space.

APPENDIX B: THE ANNULMENT OF THE EVANESCENT FIELD COMPONENTS

In order to provide the necessary comparisons with the source-free case, the integrations in Eq. (A5) shall be carried out using the angular spectrum superposition. Such an approach enhances our understanding of the evanescent fields associated with the FWM aperture. The introduction of the α variable, through the transformation given in Eq. (2.10), reduces Eq. (A5) to the following form:

$$\Psi_{\rm ap}(\rho,z,t)=\frac{a_1}{8\pi\beta}\int_0^\infty d\kappa\int_{D^+}d\alpha\ \kappa^2\,\frac{\cos\alpha}{|\cos\alpha|}\sin\alpha e^{-(\kappa^2/4\beta)a\sin^2\alpha}J_0(\kappa\rho\sin\alpha)$$
$$\times\left\{\delta\left(\kappa-\frac{2\beta}{(1-\cos\alpha)}\right)+\delta\left(\kappa-\frac{2\beta}{(1+\cos\alpha)}\right)\right\}e^{i\kappa ct}e^{-i\kappa\cos\alpha\, z}. \tag{B1}$$

The contour of integration acquires one of the two forms shown in Fig. 6. Both contours share the same D_A^+ part. The choice between D_{01}^+ and D_{02}^+ depends on the complex roots of the δ function when $(\omega/c)<\chi$. These are directly related to the complex roots of $k_z \equiv \sqrt{(\omega/c)^2-\lambda^2}$, as in Eq. (A1). In this case, the bracketed term in Eq. (4.8) should pick up only the outgoing wave components in a fashion similar to that leading to Eq. (A3).

The field generated by the aperture given in Eq. (B1) can be expressed as a summation of two components,

$$\Psi_{\rm ap}(\mathbf{r},t)=\Psi_A(\mathbf{r},t)+\Psi_0(\mathbf{r},t), \tag{B2a}$$

where the aperture radiated field $\Psi_A(\mathbf{r},t)$ and the evanescent component $\Psi_0(\mathbf{r},t)$ are

$$\Psi_A(\mathbf{r},t)=\Psi_{\rm ap}(\mathbf{r},t;D_A^+) \quad \text{and} \quad \Psi_0(\mathbf{r},t)=\Psi_{\rm ap}(\mathbf{r},t;D_0^+). \tag{B2b}$$

Here $\Psi_{\rm ap}(\mathbf{r},t;D_A^+)$ is given by Eq. (B1), but with D^+ replaced by D_A^+, while $\Psi_{\rm ap}(\mathbf{r},t;D_0^+)$ has the D_0^+ contour instead of D^+. Integrating over κ in Eq. (B2), the field radiated from the aperture becomes

$$\Psi_A(\mathbf{r},t)=\Psi^{(+)}(\mathbf{r},t)+\Psi^{(-)}(\rho,-z,t). \tag{B3}$$

The first term on the right-hand side is the Whittaker field component $\Psi^{(+)}(\mathbf{r},t)$ given in Eq. (2.7). The second term is just the $\Psi^{(-)}(\mathbf{r},t)$ Whittaker component of the source-free FWM, but with z replaced by $-z$. Such a result asserts that all the wave components are traveling away from the aperture in a causal sense. Furthermore, it should be noted that the radiated field given in Eq. (B3) is exactly equal to $\Psi_{\rm ap}(\mathbf{r},t)$ derived in Eq. (4.9).

From the preceding analysis, it follows that $\Psi_{\rm ap}(\mathbf{r},t)=\Psi_A(\mathbf{r},t)$, hence one expects that $\Psi_0(\mathbf{r},t)$, which represents the evanescent field components, is identically equal to zero. We shall go, however, through the details of such a proof to highlight the differences with the case of the source-free field. For the source-free FWM, it has been shown in Sec. II that the Weyl converging and diverging evanescent fields cancel each other identically. In contradistinction, the FWM aperture generates *only* fields that are causally diverging from the aperture. In fact, the condition of the forward illumination of such an aperture is responsible for the annulment of the evanescent fields. For the contour D_0^+, $\alpha=(\pi/2)+i\alpha_I$, $\cos\alpha=-i\sinh\alpha_I$ and $\sin\alpha=\cosh\alpha_I$. Substituting in Eq. (B2),

$$\begin{aligned}\Psi_0(\mathbf{r},t)=&-i\,\frac{a_1}{8\pi\beta}\int_0^\infty d\kappa\int_{D_0^+}d\alpha_I\,\kappa^2\cosh\alpha_I\,e^{-(\kappa^2/4\beta)a_1\cosh^2\alpha_I}J_0(\kappa\rho\cosh\alpha_I)e^{i\kappa ct}\\&\times e^{\pm\kappa\sinh\alpha_I z}\{\delta(\kappa-2\beta(\operatorname{sech}^2\alpha_I-i\operatorname{sech}\alpha_I\tanh\alpha_I))\\&+\delta(\kappa-2\beta(\operatorname{sech}^2\alpha_I+i\operatorname{sech}\alpha_I\tanh\alpha_I))\}.\end{aligned} \tag{B4}$$

The choice between the contours D_{01}^+ and D_{02}^+ depends on the choice of the sign of $k_z = \sqrt{(\omega/c)^2-\chi^2} = \pm i(\omega/c)\sinh\alpha_I$, for $(\omega/c)<\chi$. One should be careful because k_z becomes complex as a consequence of $(\omega/c)=\kappa$ being complex. The above integration has a form typical of a superposition over evanescent field components. The positive (negative) sign in the argument of the term $\exp(\pm\kappa\sinh\alpha_I z)$ is chosen when the integration is carried out over the $D_{02}^+(D_{01}^+)$ contour. Since the arguments of the δ functions have become complex, an analytical continuation into the complex κ plane is needed. For the specific choice of the sign of the argument of $\exp(i\kappa ct)$, the roots of the analytical continuation of the δ functions should have a positive imaginary part. The first δ function has the required sign of the imaginary part if α_I is negative; i.e., when the integration is carried out over the D_{02}^+ contour. In contradistinction, the second δ function has a positive imaginary part when the integration is performed over the D_{01}^+ contour. The same argument can become more transparent if we return to the expansion

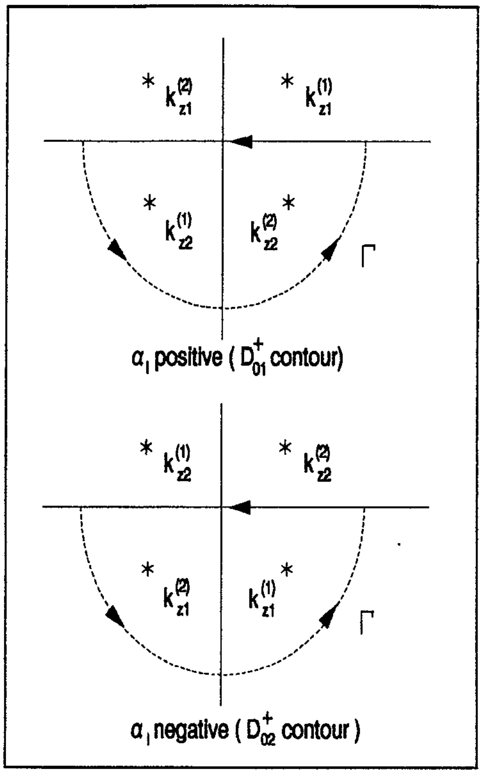

FIG. 7. The complex poles of k_z given in Eq. (B6) for the positive and negative values of α.

$$\frac{e^{-i\kappa R}}{R}=\frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} dk_z\ \lambda J_0(\lambda\rho^*)\ \frac{e^{-ik_z z}}{k_z^2+\kappa^2\ \sinh^2\ \alpha_I}, \tag{B5}$$

which is the counterpart of Eq. (A1) if α becomes complex. The poles of k_z are $k_{z1}=i\kappa$ sinh α_I and $k_{z2}=-i\kappa$ sinh α_I. One should notice, however, that the two δ functions in Eq. (B4) lead to complex values of κ. If this fact is taken into consideration, we end up with four poles; specifically

$$k_{z1}^{(1)}=2\beta\ \tanh\ \alpha_I(\tanh\ \alpha_I+i\ \text{sech}\ \alpha_I),\quad k_{z2}^{(1)}=-2\beta\ \tanh\ \alpha_I(\tanh\ \alpha_I+i\ \text{sech}\ \alpha_I),$$

$$k_{z1}^{(2)}=-2\beta\ \tanh\ \alpha_I(\tanh\ \alpha_I-i\ \text{sech}\ \alpha_I)\quad \text{and}\quad k_{z2}^{(2)}=2\beta\ \tanh\ \alpha_I(\tanh\ \alpha_I-i\ \text{sech}\ \alpha_I). \tag{B6}$$

The first two poles correspond to the roots of the first δ function in Eq. (B4), while the other two correspond to the second one. The positions of such poles in the complex k_z plane are shown in Fig. 7 for the two cases, when α_I is positive (the contour D_{01}^+ is used) and α_I is negative (the contour D_{02}^+ is used). The poles in the upper half of the k_z plane do not contribute to the integration over k_z in Eq. (B5). Furthermore, the causality condition enforced by the bracketed term in Eqs. (4.8) and (A3) leads solely to the contribution of the poles having a positive real part. Thus, the first δ function in Eq. (B4) contributes to the integration only if D_{02}^+ is used. The second δ function needs α_I to be positive to contribute to the integration, i.e., the D_{01}^+ contour has to be used. The former picks up the pole $k_{z1}^{(1)}$, while the latter selects $k_{z2}^{(2)}$. With such points taken into consideration, the integration in Eq. (B4) is carried out over κ to give

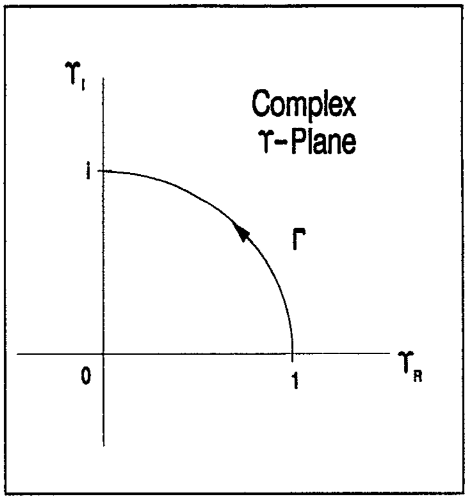

FIG. 8. The Γ contour of integration in the complex Υ plane used in Eq. (B8).

$$\begin{aligned}\Psi_0(\mathbf{r},t)=&-\frac{i\beta a_1}{2\pi}\int_0^{-\infty}d\alpha_I\ \text{sech}\ \alpha_I(\text{sech}\ \alpha_I-i\ \tanh\ \alpha_I)^2e^{-\beta a_1(\text{sech}\ \alpha_I-i\ \tanh\ \alpha_I)^2}\\&\times J_0(2\beta\rho(\text{sech}\ \alpha_I-i\ \tanh\ \alpha_I))e^{i2\beta ct(\text{sech}\ \alpha_I-i\tanh\ \alpha_I)\text{sech}\ \alpha_I}e^{+2\beta z(\text{sech}\ \alpha_I-i\ \tanh\ \alpha_I)\tanh\ \alpha_I}\\&-\frac{i\beta a_1}{2\pi}\int_0^{\infty}d\alpha_I\ \text{sech}\ \alpha_I(\text{sech}\ \alpha_I+i\ \tanh\ \alpha_I)^2e^{-\beta a_1(\text{sech}\ \alpha_I+i\ \tanh\ \alpha_I)^2}\\&\times J_0(2\beta\rho(\text{sech}\ \alpha_I+i\ \tanh\ \alpha_I))e^{i2\beta ct(\text{sech}\ \alpha_I+i\ \tanh\ \alpha_I)\text{sech}\ \alpha_I}e^{-2\beta z(\text{sech}\ \alpha_I+i\ \tanh\ \alpha_I)\tanh\ \alpha_I}.\end{aligned}\tag{B7}$$

Next, we introduce the new complex variable $\Upsilon=\text{sech}\ \alpha_I\mp i\ \tanh\ \alpha_I$, where the upper sign is used for the first integration, while the lower sign corresponds to the second one. This change of variables reduces Eq. (B7) to

$$\begin{aligned}\Psi_0(\mathbf{r},t)=&-\frac{\beta a_1}{2\pi}\int_\Gamma d\Upsilon\ \Upsilon J_0(2\beta\rho\Upsilon)e^{-\beta a_1\Upsilon^2}e^{+i(\Upsilon^2+1)\beta ct}e^{-i(\Upsilon^2-1)\beta z}\\&+\frac{\beta a_1}{2\pi}\int_\Gamma d\Upsilon\ \Upsilon J_0(2\beta\rho\Upsilon)e^{-\beta a_1\Upsilon^2}e^{+i(\Upsilon^2+1)\beta ct}e^{-i(\Upsilon^2-1)\beta z}=0.\end{aligned}\tag{B8}$$

Both integrations are carried out in the complex Υ plane, and they share the same contour of integration shown in Fig. 8. Consequently, the field $\Psi_0(\mathbf{r},t)$ that results from superimposing the evanescent components is identically equal to zero.

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