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The propagating and evanescent field components of localized wave solutions

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Abstract

The diverging and converging components of localized wave solutions are studied within the framework of both the Whittaker and Weyl plane wave expansions. The specific example of the splash pulse is considered because its evanescent components could be derived in an explicit closed form. It is shown that, in the Weyl picture, the evanescent fields associated with the diverging and converging components of the splash pulse cancel each other identically. The splash pulse is, hence, composed solely of backward and forward propagating components of the Whittaker type.

1. Introduction

A large number of localized wave solutions have been reported recently [1–11]. It has been demonstrated both theoretically and experimentally that such solutions exhibit extended ranges of localization. Experiments [12,13] verifying the propagation of such localized waves utilize independently addressable, finite-sized arrays. The resulting fields approximate to a great extent the forward propagating components of the theoretically predicted solutions [14].

It is well known, that the aforementioned localized wave solutions are composed of forward and backward propagating components. Such a behavior is reminiscent of the source-free focus wave mode (FWM) solution [15]. The latter has been criticized for being dominated by acausal incoming components [16,17]. Nonetheless, the analysis adopted in such a study of the causality of the FWMs has been restricted to a special case. It can be easily demonstrated that under a different condition (not considered by Ref. [17]) the FWM solution is composed predominantly of forward propagating components [3]. The FWM solution, nevertheless, has infinite energy content and can be only generated from an infinite aperture. To be able to utilize an aperture of a finite size, localized wave solutions of finite energy had to be deduced. In order to derive such wave solutions, various approaches have been suggested [1,3]. Most of these lead, however, to finite energy superpositions of the original focus wave modes. It has been, also, pointed out [18] that the FWM solution does not contain any evanescent wave components. One should then wonder if the finite energy solutions composed as a superposition of the FWMs can have evanescent waves associated with them.

The two aforementioned objections to the realizability of the FWM fields have been unjustly extended to criticize all other finite energy LW solutions. In most cases there have been no clear efforts to distinguish between the original

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source-free LW solutions and their counterparts which are generated causally from appropriate apertures [14]. The origin of such a confusion exists primarily because of the bidirectional nature [3] of the original source-free LW fields. We believe, however, that very good approximations to the source-free solutions can be generated through a proper choice of the parameters characterizing the LW fields. Furthermore, it is the aim of this paper to demonstrate that there exists a systematic mathematical method to derive the form of the outgoing field generated by a finite aperture from the corresponding finite-energy source-free LW solution.

Traditionally, solutions to the three dimensional wave equation can be derived in two distinct fashions. One is due to Whittaker [19] in which he uses a superposition of homogeneous plane waves propagating in opposite directions. The other has been used by Weyl [19] to express the fields outside the source region as a combination of propagating homogeneous plane waves, together with the associated inhomogeneous evanescent waves. There have been several attempts to further our understanding of the relationship between these two distinct representations [20–22]. It has been claimed that the portion of the field represented as an expansion in terms of the homogeneous backward propagating plane waves is equivalent to that expressed as a superposition of the inhomogeneous evanescent modes [21,22]. The physical meaning of such equivalence and its temporal and spatial domains of validity is not fully comprehended. Working with source-free solutions which are bidirectional [3] may complicate any implied equivalence between the backward propagating and the evanescent components. In this work, we try to use both approaches to derive finite energy FWM-like solutions. In this way, we can further our comprehension of the relationship between the Whittaker and the Weyl representations. At the same time, a deeper understanding of the inhomogeneous evanescent field content of the FWM-like solutions can be achieved. Even though it is already known that for any source-free field there are no evanescent components associated with it, nevertheless, to separate the Weyl diverging field provides a rigorous and a systematic method to deduce the physically realizable outgoing LW field generated from an aperture. This is of an utmost importance because the performance of such an outgoing field is the one that should be investigated and any judgement of the LW solutions should be deduced from the behavior of their outgoing Weyl component. In what follows we shall concentrate on a specific finite energy solution; namely the splash modes [1]. Such a wave solution has been chosen because of the ease by which the calculations could be performed in both the Whittaker and the Weyl representations.

2. Fourier composition of the splash pulse

In this section, we shall deal with the specific example of the splash pulse. We do so because of the simplicity by which the mathematics could be handled. It will be shown in the next section that the diverging, converging and evanescent components of such a solution can all be derived explicitly in closed form. The splash modes were originally [1] deduced as a superposition over the FWM solutions. Similarly, they could be synthesized from a bidirectional representation [3] that provides the most suitable basis for these kinds of waves. A simple transformation links the bidirectional representation to the Fourier one. In the Fourier picture, the Fourier superposition leading to the splash pulse solution [3] can be written as

$$\Psi(\rho, z, t) = \frac{1}{(2\pi)^2} \int_0^\infty d\chi \int_0^\infty d\omega \int_{-\infty}^\infty dk_z \left(\frac{\pi}{c} \exp\{ -a_1[(\omega/c)+k_z]/2\} \exp\{ -a_2[(\omega/c)-k_z]/2\} \right) \chi J_0(\chi\rho)$$
$$\times \exp(-ik_z z) \exp(+i\omega t) \, \delta((\omega/c)^2 - k_z^2 - \chi^2) \,. \tag{2.1}$$

The above integration may be carried out in two distinct fashions, each leading to a different representation. In particular, one can integrate over ω first, thus, ending up with a Whittaker type of expansion. In contradistinction, an integration over k_z first leads to the Weyl superposition over homogeneous and inhomogeneous plane waves. Using the former approach, we integrate over ω first to obtain

$$\Psi(\rho, z, t) = \frac{1}{(2\pi)^2} \int_0^\infty d\chi \int_{-\infty}^\infty dk_z \frac{\pi c}{2\omega_+} \exp[-k_z(a_1-a_2)/2]$$

$$\times \exp[-\omega_+(a_1+a_2)/2c]\ \chi J_0(\chi\rho) \exp(-ik_z z) \exp(+i\omega_+ t) .$$

Here $\omega_+ = c\sqrt{k_z^2+\chi^2}$. The integration over k_z can be divided into two components. One traveling in the positive z-direction, while the other is propagating in the negative z-direction; specifically,

$$\Psi(\boldsymbol{r}, t) = \Psi^{(+)}(\boldsymbol{r}, t) + \Psi^{(-)}(\boldsymbol{r}, t) . \tag{2.2}$$

The positive and negative components are given explicitly by the following integrals

$$\Psi^{(+)}(\boldsymbol{r}, t) = \frac{1}{8\pi} \int_0^\infty d\chi \int_0^\infty dk_z \frac{c}{\omega_+} \exp[-k_z(a_1-a_2)/2]$$

$$\times \exp[-\omega_+(a_1+a_2)/2c]\ \chi J_0(\chi\rho) \exp(-ik_z z) \exp(+i\omega_+ t) , \tag{2.3a}$$

and

$$\Psi^{(-)}(\boldsymbol{r}, t) = \frac{1}{8\pi} \int_0^\infty d\chi \int_0^\infty dk_z \frac{c}{\omega_+} \exp[+k_z(a_1-a_2)/2]$$

$$\times \exp[-\omega_+(a_1+a_2)/2c]\ \chi J_0(\chi\rho) \exp(+ik_z z) \exp(+i\omega_+ t) . \tag{2.3b}$$

The above two integrals can be easily evaluated by reversing the order of integration, introducing the change of variables $(k_z^2+\chi^2)^{1/2} = s$, and using formula (6.616.2) in Gradshteyn and Ryzhik [23] to obtain

$$\Psi^{(\pm)}(\boldsymbol{r}, t) = \frac{1}{8\pi(p^2-q^2+\rho^2)} \left(1 \mp \frac{q}{\sqrt{p^2+\rho^2}}\right) . \tag{2.4}$$

Here $p = [((a_1+a_2)/2) - ict]$ and $q = [((a_1-a_2)/2) + iz]$. It can be deduced, from the above expression, that the forward propagating component $\Psi^{(+)}(\boldsymbol{r}, t)$ is larger than the backward propagating one $\Psi^{(-)}(\boldsymbol{r}, t)$ when q is negative. Recalling that $q = [((a_1-a_2)/2) + iz]$, then a necessary condition for the predominance of $\Psi^{(+)}(\boldsymbol{r}, t)$ over $\Psi^{(-)}(\boldsymbol{r}, t)$ is that $a_2 \gg a_1$. This is the same result that has been obtained earlier [3] from comparing the Fourier spectral content of the two components. Since the total field of the splash mode in Eq. (2.2) is the sum of the forward and backward propagating wave components, then Eq. (2.4) yields

$$\Psi(\boldsymbol{r}, t) = \frac{1}{4\pi(p^2-q^2+\rho^2)} , \tag{2.5}$$

which is the known splash mode solution [1]. The procedure followed above is akin to that of the Whittaker representation. The final solution is an expansion in terms of positive and negative going plane waves.

To arrive at the Weyl picture we have to start by integrating Eq. (2.1) over the k_z variable. Since k_z can have both positive and negative values, the integration may be split into two parts, viz.,

$$\Psi(\boldsymbol{r}, t) = \Psi^{(d)}(\boldsymbol{r}, t) + \Psi^{(c)}(\boldsymbol{r}, t) , \tag{2.6}$$

where $\Psi^{(d)}(\boldsymbol{r}, t)$ and $\Psi^{(c)}(\boldsymbol{r}, t)$ correspond to waves diverging and converging on an aperture situated at $z=0$. In what follows, we shall be only interested in the Weyl expansion associated with the positive z half space. The two components in Eq. (2.6) can be written explicitly as

$$\Psi^{(d)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_0^\infty \mathrm{d}\chi\,\chi J_0(\chi\rho)\int_0^\infty \mathrm{d}(\omega/c)\,\frac{\exp[-p(\omega/c)]}{\sqrt{(\omega/c)^2-\chi^2}}\exp[-q\sqrt{(\omega/c)^2-\chi^2}\,]\,, \tag{2.7a}$$

and

$$\Psi^{(c)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_0^\infty \mathrm{d}\chi\,\chi J_0(\chi\rho)\int_0^\infty \mathrm{d}(\omega/c)\,\frac{\exp[-p(\omega/c)]}{\sqrt{(\omega/c)^2-\chi^2}}\exp[+q\sqrt{(\omega/c)^2-\chi^2}\,]\,. \tag{2.7b}$$

These two equations differ from Eqs. (2.3a) and (2.3b) by the possibility that the square roots in the integrands can become imaginary if $\chi>(\omega/c)$. Thus, with a proper choice of the sign of the imaginary square root, the corresponding portions of the above integrations become superpositions of exponentially decaying evanescent components in the positive z half space. For $\chi<(\omega/c)$, the integration in Eq. (2.7a) is comprised of plane waves moving in the positive z direction. Such wave components can be viewed as outgoing from an aperture situated at $z=0$. In a similar fashion, Eq. (2.7b) represents a superposition of incoming plane waves converging on the same aperture. To separate the evanescent portions of the fields represented in Eqs. (2.7) from their propagating components, it is preferable to transform the integrals into the corresponding angular spectral superpositions [24].

3. The angular spectrum of the splash pulse

The angular spectral content is usually expressed as a superposition over the spherical angles α and β of the propagation vector $\boldsymbol{k}$ [24]. The Fourier spectrum of the splash pulse is azimuthally symmetric [cf. Eq. (2.1)], hence, the corresponding angular spectrum is independent of β. In fact, the angular superposition of the splash pulse can be directly obtained from Eqs. (2.3) and (2.7) by simple changes of variables. In what follows, we rederive the expression given in Eq. (2.4) using the angular spectrum superposition. This would be beneficial when we attempt to deduce equivalent expressions for the diverging and converging components of the Weyl representation given in Eqs. (2.6) and (2.7). Starting with Eqs. (2.3), we introduce the new variables

$$\chi=\kappa\sin\alpha \quad\text{and}\quad k_z=\kappa\cos\alpha\,. \tag{3.1}$$

Such a change of variables transforms the integrals in Eqs. (2.3a) and (2.3b) into the following form

$$\Psi^{(+)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_0^{\pi/2}\mathrm{d}\alpha\int_0^\infty \mathrm{d}\kappa\,\kappa\sin\alpha\,J_0(\kappa\rho\sin\alpha)$$
$$\times\exp\{-\kappa[(a_1+a_2)+(a_1-a_2)\cos\alpha]/2\}\exp[-\mathrm{i}\kappa(z\cos\alpha-ct)]\,, \tag{3.2a}$$

and

$$\Psi^{(-)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_{\pi/2}^{\pi}\mathrm{d}\alpha\int_0^\infty \mathrm{d}\kappa\,\kappa\sin\alpha\,J_0(\kappa\rho\sin\alpha)$$
$$\times\exp\{-\kappa[(a_1+a_2)+(a_1-a_2)\cos\alpha]/2\}\exp\{-\mathrm{i}\kappa(z\cos\alpha-ct)\}\,. \tag{3.2b}$$

The integrands in the two Eqs. (3.2a) and (3.2b) are identical. The only difference between the positive traveling component $\Psi^{(+)}(\boldsymbol{r},t)$ and the negative traveling one $\Psi^{(-)}(\boldsymbol{r},t)$ is in the limits of the integration over α. Notice that $\cos\alpha$ takes positive (negative) values for $0\le\alpha<\pi/2$ ($\pi/2<\alpha\le\pi$), thus, resulting in a superposition over plane waves traveling in the positive (negative) z-direction. Using Eq. (6.623.2) in Gradshteyn and Ryzhik [23], the integrations over κ can be evaluated. Subsequently, the change of variables $\lambda=\cos\alpha$ reduces the integration

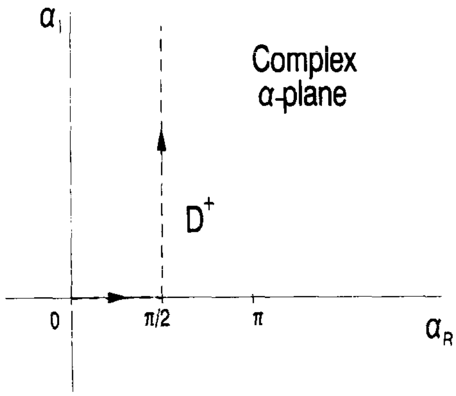

Fig. 1. The D^+ contour in the complex α-plane.

over α into an algebraic expression that could be integrated using formulae (2.264.5) and (2.264.6) in Gradshtyn and Ryzhik, to give the positive and negative going components of the Whittaker representation as in Eq. (2.4).

Next, we return to the integrations given in Eqs. (2.7a) and (2.7b), which correspond to the wave components of the Weyl representation. Since ω can be either greater or less than χc, the quantity $[(\omega/c)^2-\chi^2]^{1/2}$ might become imaginary. Thus, the resulting integrations have to be carried out in the complex α-plane. We start with the diverging component given in Eq. (2.7a). The transformations

$$\chi=\kappa \sin\alpha, \quad \text{and} \quad \omega/c=\kappa \quad \text{with} \quad \sqrt{(\omega/c)^2-\chi^2}=\kappa\cos\alpha \tag{3.3}$$

reduce Eq. (2.7a) to the following form

$$\Psi^{(d)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_{D^+}\mathrm{d}\alpha\int_0^\infty \mathrm{d}\kappa\,\kappa\sin\alpha\,J_0(\kappa\rho\sin\alpha)$$
$$\times\exp\{-\kappa[(a_1+a_2)+(a_1-a_2)\cos\alpha]/2\}\exp[-\mathrm{i}\kappa(z\cos\alpha-ct)]\,. \tag{3.4}$$

The contour D^+, shown in Fig. 1, is chosen in the complex α-plane for $z>0$. From the transformation relationships (3.3), it is clear that α is real as long as $\chi\le\kappa$. Hence, as α takes real values between 0 and $\pi/2$, the value of $\cos\alpha$ ranges from 1 to 0. The integration in Eq. (3.4) is, thus, a superposition over plane waves propagating away from the aperture into the positive z half space. In general, the angle α is complex with $\sin(\alpha_R+\mathrm{i}\alpha_I)=\chi/\kappa$. For $\chi\le\kappa$, the first portion of D^+ has $\alpha=\alpha_R$ and $\alpha_I=0$, where $0\le\alpha_R<\pi/2$. The second portion of the contour D^+ corresponding to $\chi\ge\kappa$ has $\alpha_R=\pi/2$, for which $\sin\alpha=\sin((\pi/2)+\mathrm{i}\alpha_I)=\cosh\alpha_I$ and $\cos\alpha=-\mathrm{i}\sinh\alpha_I$. The imaginary part α_I has been chosen to take only positive values between 0 and ∞, such that $\sinh\alpha_I$ stays positive. Consequently, the exponential dependence in the integrand $\exp(-\mathrm{i}\kappa z\cos\alpha)=\exp(-\kappa z\sinh\alpha_I)$ decays to 0 as $z\to\infty$. These exponential functions do not represent propagating wave components, but inhomogeneous evanescent modes. The integration in Eq. (3.4) can now be split into two parts, one represented by a superposition of outgoing waves and the other is an expansion in terms of the associated evanescent modes. Specifically, we have

$$\Psi^{(d)}(\boldsymbol{r},t)=\Psi_1^{(d)}(\boldsymbol{r},t)+\Psi_2^{(d)}(\boldsymbol{r},t)\,, \tag{3.5a}$$

where

$$\Psi_1^{(d)}(\boldsymbol{r},t)=\frac{1}{8\pi}\int_0^{\pi/2}\mathrm{d}\alpha_R\int_0^\infty \mathrm{d}\kappa\,\kappa\sin\alpha_R\,J_0(\kappa\rho\sin\alpha_R)$$
$$\times\exp\{-\kappa[(a_1+a_2)+(a_1-a_2)\cos\alpha_R]/2\}\exp[-\mathrm{i}\kappa(z\cos\alpha_R-ct)]\,, \tag{3.5b}$$

and

$$\Psi_2^{(d)}(\boldsymbol{r},t)=\frac{\mathrm{i}}{8\pi}\int\limits_0^\infty \mathrm{d}\alpha_\mathrm{I}\int\limits_0^\infty \mathrm{d}\kappa\,\kappa\cosh\alpha_\mathrm{I}\,J_0(\kappa\rho\cosh\alpha_\mathrm{I})$$

$$\times\exp\{-\kappa[(a_1+a_2)-\mathrm{i}(a_1-a_2)\sinh\alpha_\mathrm{I}]/2\}\exp(-\kappa\sinh\alpha_\mathrm{I}z)\exp(\mathrm{i}\kappa ct)\,. \tag{3.5c}$$

The first integral is identically equal to the forward going component of the Whittaker representation given in Eq. (3.2a), therefore, we have

$$\Psi_1^{(d)}(\boldsymbol{r},t)=\Psi^{(+)}(\boldsymbol{r},t)=\frac{1}{8\pi(p^2-q^2+\rho^2)}\left(1-\frac{q}{\sqrt{p^2+\rho^2}}\right). \tag{3.6}$$

As for the second portion of the integration over the contour D^+, Eq. (3.5c) can be integrated over κ by making use of formula (6.623.2) in Gradshteyn and Ryzhik [23] to obtain

$$\Psi_2^{(d)}(\boldsymbol{r},t)=\frac{\mathrm{i}}{8\pi}\int\limits_0^\infty \mathrm{d}\alpha_\mathrm{I}\frac{\cosh\alpha_\mathrm{I}(p-\mathrm{i}q\sinh\alpha_\mathrm{I})}{[(p-\mathrm{i}q\sinh\alpha_\mathrm{I})^2+(\rho\cosh\alpha_\mathrm{I})^2]^{3/2}}. \tag{3.7}$$

With the new variable, $\lambda=\sinh\alpha_\mathrm{I}$ the integration in Eq. (3.7) may be evaluated by using the identities (2.264.5) and (2.264.6) in Gradshteyn and Ryzhik. After substituting for the appropriate limits, we get

$$\Psi_2^{(d)}(\boldsymbol{r},t)=\frac{1}{8\pi(p^2-q^2+\rho^2)}\left(\frac{p}{\sqrt{q^2-\rho^2}}+\frac{q}{\sqrt{p^2+\rho^2}}\right), \tag{3.8}$$

which is not equal to $\Psi^{(-)}(\boldsymbol{r},t)$ given in Eq. (2.4). Thus, the homogeneous backward propagating component of the Whittaker representation is not equivalent to the inhomogeneous evanescent field associated with the radiation diverging from an aperture in the Weyl picture. Summing up the components in Eqs. (3.6) and (3.8) one obtains the total field

$$\Psi^{(d)}(\boldsymbol{r},t)=\frac{1}{8\pi(p^2-q^2+\rho^2)}\left(1+\frac{p}{\sqrt{q^2-\rho^2}}\right), \tag{3.9}$$

diverging from an aperture situated at $z=0$ into the half space $z>0$. Such a wave function differs from the splash pulse solution written in Eq. (2.5) by an amount which should be equal to $\Psi^{(c)}(\boldsymbol{r},t)$, the Weyl field component converging on the aperture placed at $z=0$. To verify such a claim, we should go back and evaluate the integration (2.7b). The change of variables

$$\chi=\kappa\sin\alpha,\quad\text{and}\quad\omega/c=\kappa,\quad\text{with }\sqrt{(\omega/c)^2-\chi^2}=-\kappa\cos\alpha \tag{3.10}$$

transforms Eq. (2.7b) to the following form

$$\Psi^{(c)}(\boldsymbol{r},t)=\frac{-1}{8\pi}\int\limits_{C^+}\mathrm{d}\alpha\int\limits_0^\infty \mathrm{d}\kappa\,\kappa\sin\alpha\,J_0(\kappa\rho\sin\alpha)$$

$$\times\exp\{-\kappa[(a_1+a_2)+(a_1-a_2)\cos\alpha]/2\}\exp[-\mathrm{i}\kappa(z\cos\alpha-ct)]\,. \tag{3.11}$$

Here, the contour C^+ shown in Fig. 2 corresponds to a superposition of incoming or converging waves from the $z>0$ half space on an aperture situated at $z=0$. It should be noted that the integration in Eq. (3.11) resembles that in Eq. (3.4) except for being negative and having the contour C^+ instead of D^+. The choice of the former contour is dictated by the need that $\cos\alpha$ be negative when $\kappa>\chi$. So, as χ changes from 0 to κ, the real part of the angle α

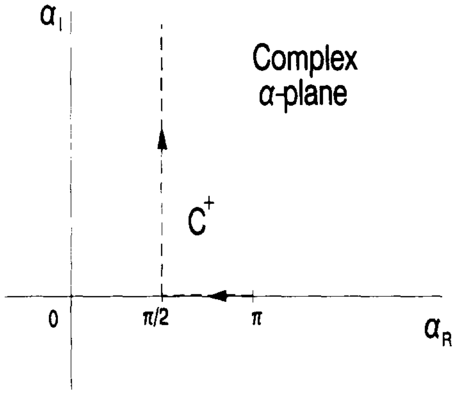

Fig. 2. The C^+ contour in the complex α-plane.

goes between π and $\pi/2$. The value of $\sin\alpha$ stays positive, while $\cos\alpha$ becomes negative with values ranging from -1 to 0. The integration in Eq. (3.11) is, thus, a superposition over plane waves propagating towards the aperture from the positive z half space. When $\chi > \kappa$, the angle α becomes complex with $\sin(\alpha_R + i\alpha_I) = \chi/\kappa$. For the second portion of the contour C^+ one has $\alpha_R = \pi/2$, hence, $\sin\alpha = \sin((\pi/2) + i\alpha_I) = \cosh\alpha_I$ and $\cos\alpha = -i\sinh\alpha_I$. The imaginary part α_I has been chosen to take only positive values between 0 and ∞, such that $\sinh\alpha_I$ stays positive. As in the case of the Weyl diverging components, the preceding choice of the contour ensures that the exponential dependence in the integrand $\exp(-i\kappa z\cos\alpha) = \exp(-\kappa z\sinh\alpha_I)$ decays to 0 as $z\to\infty$. Furthermore, the integration in Eq. (3.11) can now be split into two parts, one representing a superposition of incoming wave components and the other is an expansion in terms of the exponentially decaying evanescent modes. Specifically, we have

$$\Psi^{(c)}(r,t) = \Psi_1^{(c)}(r,t) + \Psi_2^{(c)}(r,t)\,, \tag{3.12}$$

where the $\Psi_1^{(c)}(r,t)$ and $\Psi_2^{(c)}(r,t)$ correspond to the propagating and evanescent components, respectively. The analysis that follows is identical to the case of the diverging Weyl component to arrive at the final expressions:

$$\Psi_1^{(c)}(r,t) = \Psi^{(-)}(r,t) = \frac{1}{8\pi(p^2-q^2+\rho^2)}\left(1+\frac{q}{\sqrt{p^2+\rho^2}}\right), \tag{3.13}$$

$$\Psi_2^{(c)}(r,t) = -\Psi_2^{(d)}(r,t) = \frac{-1}{8\pi(p^2-q^2+\rho^2)}\left(\frac{p}{\sqrt{q^2-\rho^2}}+\frac{q}{\sqrt{p^2+\rho^2}}\right). \tag{3.14}$$

Summing up the components in Eqs. (3.13) and (3.14) one obtains the total Weyl converging field

$$\Psi^{(c)}(r,t) = \frac{1}{8\pi(p^2-q^2+\rho^2)}\left(1-\frac{p}{\sqrt{q^2-\rho^2}}\right). \tag{3.15}$$

The splash pulse solution in Eq. (2.5) may now be derived by summing up the diverging and converging components given in Eqs. (3.9) and (3.15). The same result could have been obtained by simply adding up the Weyl propagating components $\Psi_1^{(d)}(r,t)$ and $\Psi_1^{(c)}(r,t)$ expressed in Eqs. (3.6) and (3.13). This is the case because the inhomogeneous evanescent components cancel out identically as can be seen from summing up Eqs. (3.8) and (3.14). This means that the total evanescent field associated with the splash pulse is equal to zero. So it is not only the FWM solution that does not have evanescent modes associated with it, but even a superposition of the FWM solutions leading to a finite energy wave function might not have any evanescent fields. This point can be better understood if we refer back to contours D^+ and C^+ in the complex α-plane. Such contours have been used to evaluate $\Psi^{(d)}(r,t)$ and $\Psi^{(c)}(r,t)$ given in Eqs. (3.4) and (3.11). The integrands in the aforementioned integrations differ only by a negative sign. The difference in the sign can be accommodated into the reversal of the

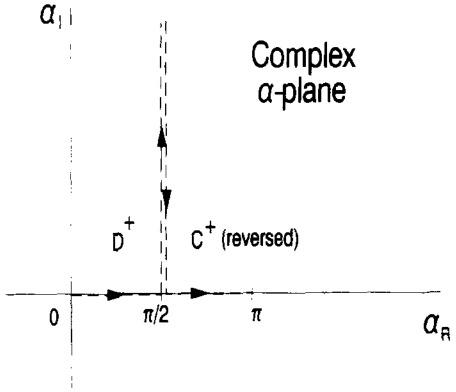

Fig. 3. The D^+ contour together with the reversed C^+ contour in the complex α-plane.

sense of the contour integration over C^+. If the two contours are plotted together, it can be seen from Fig. 3 that the contour integrations contributing to the evanescent components cancel each other. The integrations over α_I, thus, subtract from each other and we are left with the integrations over α_R ranging from $0 \to \pi$. This leads to the Whittaker superposition of forward and backward propagating components.

One can perceive the splash pulse or any other FWM-like solution as source free fields which are built from diverging and converging components. However, the associated evanescent modes cancel out identically. Similar results have been obtained in Ref. [22] in connection to spherical waves and their representations in the Whittaker and the Weyl expansions. This does not rule out the possibility of exciting approximations to the LW solutions from finite apertures. In the case of the splash pulse, the propagating field excited from an aperture placed at $z=0$ is given by $\Psi_1^{(d)}(\boldsymbol{r},t)=\Psi^{(+)}(\boldsymbol{r},t)$ while the associated evanescent field is $\Psi_2^{(d)}(\boldsymbol{r},t)$. It has been demonstrated that for localized wave solutions an appropriate choice of parameters [2,14] can ensure that most of the field energy is propagating away from the aperture. Such conclusions have, also, been confirmed by experiment [12,13].

The diverging splash pulse field $\Psi^{(d)}(\boldsymbol{r},t)$ given in Eq. (3.9) exhibits some interesting properties. If we look at the center of the pulse at $\rho=0$ and $z=ct$, we find that

$$\Psi^{(d)}(\rho=0, z=ct)=\frac{1}{4\pi(a_2-\mathrm{i}2z)\left[(a_1-a_2)+\mathrm{i}2z\right]}. \tag{3.16}$$

For forward propagation, the condition $a_2 \gg a_1$ is emphasized. Thus, the center of the pulse starts to decay at $z_R \sim (a_2/2)$. This defines the Rayleigh distance separating the near and far field regions. Notice that the decay of the center of the diverging field $\Psi^{(d)}(\boldsymbol{r},t)$ goes as $(1/z^2)$ instead of the $(1/z)$ characterizing the source free splash pulse. As such, the decay of $\Psi^{(d)}(\boldsymbol{r},t)$ beyond the Rayleigh distance is very rapid; a property required in several medical applications. On the other hand, within the Rayleigh distance $z<a_2$, the amplitude of the center does not depend on z. At $z=ct$ inside the near field region, we have $(p^2-q^2)\sim a_1a_2$, $p\sim(a_2/2)$ and $q\sim(a_1/2)$. The diverging field becomes

$$\Psi_{\mathrm{nf}}^{(d)}(\rho, z=ct)=\frac{1}{4\pi(a_1a_2+\rho^2)}\left(1+\frac{a_2}{\sqrt{a_2^2-4\rho^2}}\right). \tag{3.17}$$

For $a_2 \gg \rho$, while keeping $a_2 \gg a_1$, the expression in Eq. (3.17) reduces to

$$\Psi_{\mathrm{nf}}^{(d)}(\rho, z=ct)\simeq\frac{1}{4\pi(a_1a_2+\rho^2)}. \tag{3.18}$$

Now, if we choose $a_2=100$ cm and $a_1=0.01$ cm, we start having a $(1/\rho^2)$ roll off from the axis for $\rho>1$ cm. This means that we have a fairly localized field of constant amplitude over most of the near field region. The intensity of such a pulse rolls off the axis as $(1/\rho^4)$. Once we approach the near-far field limit, such a field decays very rapidly. This kind of behavior is desirable for several medical applications, like ultrasound imaging and designing hyperthermia applicators.

4. Conclusions

In this work, we have shown that the Whittaker and the Weyl representations of the splash pulse are equivalent. The former uses a superposition of outgoing and incoming plane waves, while the latter is composed of plane waves diverging and converging from an aperture together with the associated evanescent components. It also has been demonstrated that the Whittaker negative propagating waves are not equivalent to the evanescent fields associated with its Weyl diverging component. Subsequently, in order that the two representations arrive at the same source-free splash pulse solution, we need to take into consideration the converging Weyl propagating and evanescent components. The latter add up to the Weyl diverging fields in such a way that the evanescent modes cancel out identically. Thus, it is established that finite energy solutions which are a weighted superposition of the focus wave modes might not have evanescent fields associated with them. This property is shared by the original FWM solution.

Because FWM-like solutions are composed of forward and backward propagating components, it has been argued [17] that this results in "grave" problems with causality and the possibility of exciting such pulses from real sources. The analysis carried out in this paper has separated the components diverging out of an aperture from waves converging on it. It has, also, been shown that with a proper choice of parameters (viz. $a_2 \gg a_1$), most of the energy of the splash pulse is contributed to the forward propagating component. The field $\Psi^{(d)}(\boldsymbol{r}, t)$ then will be a good approximation to the original splash pulse $\Psi(\boldsymbol{r}, t)$. Similar results have been obtained in association with other localized wave solutions, where it has been demonstrated that with an appropriate adjustment of parameters, the acausal components can be negligible for all practical purposes.

Finally, it should be emphasized that using the procedure presented in this paper, we can always identify the Weyl diverging and evanescent field components of any other localized wave solution. Such field components are the ones that could be realized by physical apertures. In most cases, it is difficult to derive a closed-form expression for the Weyl diverging field. Nevertheless, under specific choice of parameters $\Psi^{(d)}(\boldsymbol{r}, t)$ can be made *almost* equal to the source-free solution $\Psi(\boldsymbol{r}, t)$. The field excited from a physical aperture should then resemble the source-free LW solution for all ranges of interest. In such cases, it is not apparent what "grave" problems could be associated with the causal excitation of the LW fields. Extracting the Weyl diverging field components from the source-free LW solutions, thus, provides a rigorous avenue for deriving the launchable portion of the original FWM-like solutions. This provides a solid basis for any future debates on the launchibility of the various LW solutions.

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