A novel approach to the synthesis of nondispersive wave packet solutions to the Klein–Gordon and Dirac equations

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A systematic approach to the derivation of exact nondispersive packet solutions to equations modeling relativistic massive particles is introduced. It is based on a novel bidirectional representation used to synthesize localized Brittingham-like solutions to the wave and Maxwell's equations. The theory is applied first to the Klein–Gordon equation; the resulting nondispersive solutions can be used as de Broglie wave packets representing localized massive scalar particles. The resemblance of such solutions to previously reported nondispersive wave packets is discussed and certain subtle aspects of the latter, especially those arising in connection to the correct choice of dispersion relationships and the definition of group velocity, are clarified. The results obtained for the Klein–Gordon equation are also used to provide nondispersive solutions to the Dirac equation which models spin 1/2 massive fermions.

I. INTRODUCTION

A large body of work has been inspired recently by Brittingham's focus wave mode (FWM) solutions[1] to Maxwell's equations. Such solutions are built up of a Gaussian envelope, traveling in one direction, multiplied by a plane wave traveling in the opposite direction. The FWMs have the appealing features that they undergo only local variations, they do not spread out as they propagate in free space, and they travel with the speed of light in straight lines. The vector FWMs were derived by Brittingham[1] in a heuristic way. More motivated derivations were carried out by Sezginer,[2] Belanger,[3] and Ziolkowski[4] who obtained FWM solutions to the scalar wave equation and used them as Hertzian potentials to determine the corresponding vector solutions to Maxwell's equations. Although the FWMs have an infinite total energy content, they still have a finite energy density, a property they share with sinusoidal plane-wave solutions.

The popular use of plane waves to represent moving particles defies our intuitive notion of particles as localized solutions to field equations. Other, localized solutions, e.g., Gaussian pulses, tend to spread out as they propagate in free space. In contradistinction, the FWM solutions have the attractive property of staying localized for all time; as a consequence, they are more suitable for representing light particles (photons). The importance of this property is quite clear in view of the fact that particle localization is the only phenomenon that links us to the microphysical world. For example, a track left by a particle in a cloud chamber or a dot left by a photon on a photographic plate are just manifestations of the localization of particles, a concept that has been undermined in the current interpretation of quantum mechanics.

These ideas concerning particle localization are not completely new; they reflect a position that was advocated by Einstein and de Broglie,[5,6] among others. In their view, a particle is perceived as a high concentration of a field governed by a partial differential equation, e.g., Maxwell's equations, the Klein–Gordon equation, etc. This highly concentrated field, or "bunch field," must remain localized and must not spread out as the particle travels in space-time. In this picture, the bunch field is incorporated in an extended wave field, thus combining the wave and the corpuscular aspects of matter. As in the case of massless particles, this interpretation of the wave-particle duality should be contrasted with Bohr's complementarity principle, whereby a particle manifests itself either in the form of a wave or in the form of a corpuscle, with both characters never being observed simultaneously.

If the idea of the bunch field is adopted, a representation of a particle in the form of a wave packet is one possibility. Until recently, however, it was believed that linear field equations cannot support continuous nonsingular wave packets that do not spread in free motion. (This is not the case for the massless FWMs and the massive nondispersive wave packets derived by MacKinnon.[7,8]) The other possibility is to use a "singularity solution" for representing the physical reality of a localized particle. Such a singular solution to a linear field equation is an approximation to a more general solution of a corresponding nonlinear equation. The nonlinearity has a larger effect near the vicinity of the singularity, where it keeps the field amplitude large but finite. One of the first attempts to incorporate such ideas was de Broglie's in connection with his theory of the "double solution."[9] Other attempts include Madelung's hydrodynamical model[10] and de Broglie's "pilot wave" theory,[9] both of which inspired Bohm[11,12] to develop the idea of the quantum po-

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tential and to use it to give a causal interpretation of quantum mechanics. A common feature of these theories is that the particle kinematics can be derived from the information incorporated in the phase of a quantum mechanical wave function $\Psi = |\Psi| e^{i\phi}$, where both $|\Psi|$ and ϕ are real and the velocity of the particle can be given as

$$u = (1/m)\nabla\phi, \tag{1}$$

a relationship known as the "guidance formula."[9] More recent developments, along the same lines, include the introduction of solitons into field theories,[13] through the study of fields modeled by nonlinear equations, e.g., the cubic Schrödinger equation, the cubic Klein–Gordon equation, the sine-Gordon equation, etc. A rather broad class of such equations has been proposed for modeling localized particles. It is not very clear, however, whether a unique set of equations could be agreed upon to represent massive particles.

It is our purpose in this exposition to investigate the possibility of using Brittingham-like linear structures to represent massive particles. There are two options that we would like to examine. The first one is to think of these nondispersive wave packets as classical billiard-like solutions. In this case the velocity of the particle is the same as the velocity of the wave packet's envelope. The other choice is to follow de Broglie and consider such solutions as quantum mechanical objects whose kinematics can be derived from their phases as in Eq. (1). Since the original FWMs are solutions to the scalar wave equation or Maxwell's equations, they represent massless particles and their envelopes travel in free space with the speed of light. In the case of a massive particle, one should find for the Klein–Gordon equation or the Dirac equation solutions analogous to the FWMs, but with their envelopes traveling at some group velocity v_g smaller than the speed of light c. A previous attempt[14] was made to find localized solutions to the Klein–Gordon equation. These solutions were approximate, with an envelope moving at a group velocity v_g very close to the speed of light c, or exact ones with an envelope traveling at the speed of light, a feature that makes them physically unattractive. A Brittingham-like solution to the massive Dirac equation has never been published before. However, Brittingham-like solutions to the massless Dirac equation and the spinor wave equation have been derived by Hillion.[15,16] Again, all these solutions have dealt with massless fields and, consequently, they have envelopes that move in straight lines with the speed of light. It is our aim in this paper to introduce a method for obtaining Brittingham-like solutions to massive fields, in particular, the massive scalar field modeled by the Klein–Gordon equation and the massive spinor field modeled by the Dirac equation. The work is based on an embedding technique that has been utilized to derive a natural basis for the synthesis of Brittingham-like solutions. This novel basis has been termed the bidirectional representation[17] because it is a superposition of elementary solutions built up of a product of two plane waves, one traveling to the left and the other to the right. Our plan is to give a brief introduction to the bidirectional representation in the next section and use it to derive the scalar FWMs. This method will be applied to the Klein–Gordon equation in Sec. III, where solutions analogous to the FWMs, but moving with a group velocity v_g, will be derived. It will be shown that a special case of such solutions is the nondispersive wave packet derived by MacKinnon.[7,8] A comparison of MacKinnon's work to ours will be carried out in Sec. IV. Nondispersive localized solutions to the Dirac equation will be derived in Sec. V and a general discussion of the results will be given in Sec. VI.

II. THE BIDIRECTIONAL REPRESENTATION

The bidirectional representation[17] was originally developed in order to provide a natural basis for synthesizing Brittingham-like solutions. In this section, we shall outline the salient features of this technique and use it to derive the scalar FWMs.

Consider the general equation

$$[\partial_t^2 + \hat{\Omega}(-i\nabla)]\Psi(\mathbf{r},t) = 0, \quad \mathbf{r}\in R^3, \quad t>0, \tag{2}$$

where $\hat{\Omega}(-i\nabla)$ is a positive, self-adjoint, possibly pseudo-differential operator, which can be decomposed as follows:

$$\hat{\Omega}(-i\nabla) = \hat{A}(-i\partial_z) + [\hat{\Omega}(-i\nabla) - \hat{A}(-i\partial_z)]$$

$$\equiv \hat{A}(-i\partial_z) + \hat{B}(-i\nabla_T, -i\partial_z). \tag{3}$$

The manner in which the operators $\hat{A}(-i\partial_z)$ and $\hat{B}(-i\nabla_T, -i\nabla_z)$ are chosen provides a great deal of flexibility; the operator $\hat{A}(-i\partial_z)$ may or may not be a natural part of $\hat{\Omega}(-i\nabla)$ and the choice of the preferred variable z is arbitrary. A splitting of the type given in (3) changes Eq. (2) to the form

$$\partial_t^2\Psi(\mathbf{r},t) + \hat{A}(-i\partial_z)\Psi(\mathbf{r},t) + \hat{B}(-i\nabla_T, -i\partial_z)\Psi(\mathbf{r},t) = 0. \tag{4}$$

We introduce, next, the Fourier transform with respect to the transverse (with respect to z) variables, viz.,

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_{R^2} d\boldsymbol{\kappa}\, \tilde{\psi}(\boldsymbol{\kappa},z,t) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}. \tag{5}$$

The spectrum $\tilde{\psi}(\boldsymbol{\kappa},z,t)$ is governed by the equation

$$\partial_t^2\tilde{\psi}(\boldsymbol{\kappa},z,t) + \hat{A}(-i\partial_z)\tilde{\psi}(\boldsymbol{\kappa},z,t) + \hat{B}(-\boldsymbol{\kappa}, -i\partial_z)\tilde{\psi}(\boldsymbol{\kappa},z,t) = 0. \tag{6}$$

In terms of new variables

$$\zeta = z - t\,\mathrm{sgn}(\alpha)\alpha^{-1}A^{1/2}(\alpha), \tag{7a}$$

$$\eta = z + t\,\mathrm{sgn}(\beta)\beta^{-1}A^{1/2}(\beta), \tag{7b}$$

an elementary solution to Eq. (6) is given by

$$\psi_e(z,t,\beta,\alpha) = e^{-i\alpha\zeta(\beta,\alpha)} e^{+i\beta\eta(\beta,\alpha)}, \tag{8}$$

provided that the following constraint is satisfied:

$$-[A(\alpha) + A(\beta) + 2\,\mathrm{sgn}(\alpha)A^{1/2}(\alpha)\,\mathrm{sgn}(\beta)A^{1/2}(\beta) - A(\beta-\alpha)] + B(\boldsymbol{\kappa},(\beta-\alpha)) \equiv K(\alpha,\beta,\boldsymbol{\kappa}) = 0. \tag{9}$$

The elementary solution given in Eq. (8) consists of a product of two plane waves traveling in opposite directions, with wave-number-dependent phase speeds equal to $\mathrm{sgn}(\alpha)\alpha^{-1}A^{1/2}(\alpha)$ and $\mathrm{sgn}(\beta)\beta^{-1}A^{1/2}(\beta)$, respectively. A general solution to Eq. (2) can be constructed from the

elementary solutions of the type given in (8) by a linear superposition; specifically,

$$\Psi(\mathbf{r},t)=\frac{1}{(2\pi)^2}\int_{R^2}d\boldsymbol{\kappa}\,e^{i\boldsymbol{\kappa}\cdot\rho}\int_{R^1}d\alpha\int_{R^1}d\beta\,C(\alpha,\beta,\boldsymbol{\kappa})$$
$$\times e^{+i\alpha\zeta(\alpha,z,t)}e^{-i\beta\eta(\beta,z,t)}\delta[K(\alpha,\beta,\boldsymbol{\kappa})]. \tag{10}$$

A detailed analysis of this representation and its relation to a Fourier superposition can be found in Ref. 17, where it was applied to various classes of equations, e.g., the 3-D scalar wave equation, the 3-D Klein–Gordon equation, and the telegraph equation. As mentioned earlier, the resulting solutions had envelopes moving with the speed of light, a property we would like to avoid in the next section.

As an example, we shall apply the bidirectional representation to the 3-D scalar wave equation, viz.,

$$\partial_t^2\Psi(\mathbf{r},t)-c^2\nabla^2\Psi(\mathbf{r},t)=0, \tag{11}$$

where $\hat{\Omega}(-i\nabla)$ is now defined as

$$\hat{\Omega}(-i\nabla)=-c^2\nabla^2. \tag{12}$$

We can choose the $\hat{A}(-i\partial_z)$ and $\hat{B}(-i\nabla_T,-i\partial_z)$ operators as follows:

$$\hat{A}(-i\partial_z)=-c^2\partial_z^2, \tag{13a}$$

$$\hat{B}(-i\nabla_T,-i\partial_z)=-c^2\nabla_T^2. \tag{13b}$$

This decomposition results in the characteristic variables

$$\zeta=z-ct \quad\text{and}\quad \eta=z+ct, \tag{14}$$

and the constraint relationship

$$K(\alpha,\beta,\boldsymbol{\kappa})\equiv-4\alpha\beta+\kappa^2=0. \tag{15}$$

Specializing the representation given in Eq. (10), an azimuthally symmetric solution to the scalar wave equation can be written explicitly as

$$\Psi(\mathbf{r},t)$$
$$=\frac{1}{(2\pi)^2}\int_0^\infty d\kappa\,\kappa J_0(\kappa\rho)\int_0^\infty d\beta\int_0^\infty d\alpha\,\delta\left(\alpha\beta-\frac{\kappa^2}{4}\right)$$
$$\times C(\alpha,\beta,\kappa)e^{-i\alpha(z-ct)}e^{i\beta(z+ct)} \tag{16}$$

or

$$\Psi(\mathbf{r},t)=\frac{1}{(2\pi)^2}\int_0^\infty d\kappa\int_0^\infty d\beta\,\frac{\kappa}{\beta}J_0(\kappa\rho)$$
$$\times C\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)e^{-i\kappa^2/4\beta}e^{i\beta\eta}, \tag{17}$$

upon carrying out the integration over α in Eq. (16).

Let us choose the spectrum

$$C(\kappa^2/4\beta,\beta,\kappa)=(\sqrt{\pi}/2)\sigma e^{-\sigma^2(\beta-\beta')^2}e^{-a_1\kappa^2/4\beta}. \tag{18}$$

Carrying out the integration over κ and β in Eq. (17) and taking the limit as $\sigma\to\infty$, we obtain the zeroth order FWM solution;[17] specifically,

$$\Psi(\mathbf{r},t)=[4\pi(a_1+i\zeta)]^{-1}e^{-\beta'\rho^2/(a_1+i\zeta)}e^{i\beta'\eta}. \tag{19}$$

It has been demonstrated by the authors[17] that for very small values of a_1 this function behaves like a localized pulse that moves in the positive z direction with speed c. Since a_1 is not dimensionless, we can use the more stringent condition $\beta' a_1\ll 1$. A good estimate of the waist of such a pulse is $(a_1/\beta')^{1/2}$; as a consequence, the condition $(a_1/\beta')^{1/2}\ll 1/\beta'$ has to be satisfied. If β' is assumed to be a characteristic wave number, with a corresponding wavelength $\lambda=2\pi/\beta'$, the condition given earlier becomes $(a_1/\beta')^{1/2}\ll\lambda$, and for $\Psi(\mathbf{r},t)$ to represent a localized light pulse, its waist must be much less than the characteristic wavelength of an extended wave structure associated with it. If, on the other hand, $\beta' a_1>1$, the plane-wave term $\exp(i\beta\eta)$ takes over and $\Psi(\mathbf{r},t)$ degenerates into a nonlocalized sinusoidal function traveling in the negative z direction.

Solutions such as the one in Eq. (19) can be very interesting when it comes to modeling the microphysical world; they are characterized, however, by infinite total energies. A superposition of FWMs, suggested by Ziolkowski,[4] yields finite energy, highly localized pulses of unusual decay patterns. These slow energy decay patterns have been confirmed experimentally,[18] and it has been shown that specific pulses, e.g., the modified power spectrum (MPS) pulse,[19] hold together for longer distances than Gaussian pulses.

III. THE KLEIN–GORDON EQUATION

In this section, we shall apply the bidirectional representation to the 3-D Klein–Gordon equation given by

$$\partial_t^2\Psi(\mathbf{r},t)-c^2\nabla^2\Psi(\mathbf{r},t)+\mu^2c^2\Psi(\mathbf{r},t)=0, \tag{20}$$

where $\mu=m_0c/\hbar$, m_0 being the rest mass and $\hbar$ is Planck's constant divided by 2π. A comparison of this equation with (2) shows that

$$\hat{\Omega}(-i\nabla)\equiv-c^2\nabla^2+\mu^2c^2. \tag{21}$$

In our previous work[14] the operator $\hat{\Omega}(-i\nabla)$ was split as follows:

$$\hat{\Omega}(-i\nabla)=\hat{A}(-i\partial_z)+\hat{B}(-i\nabla_T,-i\partial_z), \tag{22}$$

$$\hat{A}(-i\partial_z)=-c^2\partial_z^2, \tag{23a}$$

$$\hat{B}(-i\nabla_T,-i\partial_z)=-c^2\nabla_T^2+\mu^2c^2. \tag{23b}$$

This decomposition led to the characteristic variables

$$\zeta=z-ct,\quad \eta=z+ct, \tag{24}$$

and, upon superposition, to a wave packet with an envelope moving with the speed of light, exactly as in the case of massless particles.

In the following, we propose to split the operator $\hat{\Omega}(-i\nabla)$ in a more physical way so that we can obtain envelopes that move with a group velocity smaller than c; specifically,

$$\hat{\Omega}(-i\nabla)=\hat{A}(-i\partial_z)+\hat{B}(-i\nabla_T,-i\partial_z), \tag{25}$$

$$\hat{A}(-i\partial_z)=-c^2\partial_z^2+\mu^2c^2, \tag{26a}$$

$$\hat{B}(-i\nabla_T,-i\partial_z)=-c^2\nabla_T^2. \tag{26b}$$

This choice of the operators $\hat{A}$ and $\hat{B}$ gives rise to the characteristic variables

$$\zeta=z-ct\,[\mathrm{sgn}(\alpha)/\alpha](\alpha^2+\mu^2)^{1/2}, \tag{27a}$$

$$\eta=z+ct\,[\mathrm{sgn}(\beta)/\beta\,](\beta^2+\mu^2)^{1/2}, \tag{27b}$$

and the constraint relationship

$$K(\alpha,\beta,\kappa)=\kappa^2-[\mu^2+2\alpha\beta+2\,\mathrm{sgn}(\alpha)(\alpha^2+\mu^2)^{1/2}\times\mathrm{sgn}(\beta)(\beta^2+\mu^2)^{1/2}]=0. \tag{28}$$

Following the recipe given in Sec. II, a general solution to Eq. (20) can be written as follows:

$$\Psi(\mathbf{r},t)=\frac{1}{(2\pi)^2}\int_{R^2}d\kappa\, e^{-i\kappa\cdot\rho}\int_{R^1}d\alpha\int_{R^1}d\beta \times C(\alpha,\beta,\kappa)\delta[K(\alpha,\beta,\kappa)] \times\exp[-i\alpha(z-ct\,\mathrm{sgn}(\alpha)(\alpha^2+\mu^2)^{1/2}/\alpha)] \times\exp[i\beta(z+ct\,\mathrm{sgn}(\beta)(\beta^2+\mu^2)^{1/2}/\beta)]. \tag{29}$$

Analogously to the FWMs, we choose the spectrum entering into (29) as

$$C(\alpha,\beta,\kappa)=\bar{C}(\alpha,\kappa)\delta(\beta-\beta_0). \tag{30}$$

It follows, then, that

$$\Psi(\mathbf{r},t)=G(\rho,z,t) \times\exp[i\beta_0(z+ct\,\mathrm{sgn}(\beta_0)(\beta_0^2+\mu^2)^{1/2}/\beta_0)], \tag{31}$$

where

$$G(\rho,z,t)=\frac{1}{(2\pi)^2}\int_{R^2}d\kappa\, e^{i\kappa\cdot\rho}\int_{R^1}d\alpha\,\bar{C}(\alpha,\kappa)\delta[K(\alpha,\beta_0,\kappa)] \times\exp[i\alpha(z-ct\,\mathrm{sgn}(\alpha)(\alpha^2+\mu^2)^{1/2}/\alpha)]. \tag{32}$$

We can find explicit FWM-like solutions to Eq. (20) by choosing a spectrum $\bar{C}(\alpha,\kappa)$ and carrying out the integrations in Eq. (32). This is a very tedious task, however, especially when dealing with a complicated constraint relationship such as the one given in Eq. (28). Alternatively, we can find the differential equation governing $G(\rho,z,t)$ by substituting (31) into the 3-D Klein–Gordon equation. If this procedure is implemented, we obtain

$$i2\beta_0(\partial_z-v_g^{-1}\partial_t)G(\rho,z,t)+(\partial_z^2-c^{-2}\partial_t^2) \times G(\rho,z,t)+\nabla_T^2G(\rho,z,t)=0, \tag{33}$$

where v_g is a group velocity given by

$$v_g=c\beta_0/\mathrm{sgn}(\beta_0)(\beta_0^2+\mu^2)^{1/2}. \tag{34}$$

It should be noted that v_g can be derived by differentiating the angular frequency characterizing the left-going plane wave with respect to the wave number β_0.

Motivated by the ansatz leading to the FWMs in the case of the scalar wave equation and by the existence of the convection term $(\partial_z-v_g^{-1}\partial_t)G(\rho,z,t)$ in Eq. (33), we seek solutions of the form

$$G(\rho,z,t)\equiv G(\rho,\tau), \tag{35a}$$

$$\tau=\gamma(z-v_gt), \tag{35b}$$

$$\gamma=(1-v_g^2/c^2)^{-1/2}. \tag{35c}$$

Equation (33) becomes, then, a hyperbolized Schrödinger-like equation, viz.,

$$i4\beta_0\gamma\partial_\tau G(\rho,\tau)+\partial_\tau^2G(\rho,\tau)+\nabla_T^2G(\rho,\tau)=0. \tag{36}$$

It is now clear that v_g is the group velocity associated with a classical billiard-like particle represented by the enveloped of $G(\rho,\tau)$. In our previous work,[14] we obtained solutions to (36) for $\gamma\gg1$, or, equivalently, for $v_g\simeq c$. To obtain an exact solution to Eq. (36), we express $G(\rho,\tau)$ in the form

$$G(\rho,\tau)=g(\rho,\tau)e^{-i2\beta_0\gamma\tau}. \tag{37}$$

A substitution of (37) into (36) results in the Helmholtz equation:

$$\nabla_T^2g(\rho,\tau)+\partial_\tau^2g(\rho,\tau)+4\beta_0^2\gamma^2g(\rho,\tau)=0. \tag{38}$$

The steps leading to (38) are interesting by themselves since they reduce the 3-D Klein–Gordon equation, which is hyperbolic, to a 3-D Helmholtz equation, which is elliptic. More importantly, however, a solution to Eq. (38) represents an envelope that travels with a velocity v_g and retains its shape for all time. As a consequence, a large class of exact nondispersive solutions to the 3-D Klein–Gordon equation can be derived from exact solutions to the Helmholtz equation. One possible solution can be expressed in terms of the spherical Bessel functions, viz.,

$$g(\rho,\tau)=j_l(2\beta_0\gamma R)P_l^m(\tau/R)\cos(m\phi),$$

where $R=\sqrt{\rho^2+\tau^2}$, j_l is the spherical Bessel function of order l and P_l^m is the associated Legendre function. Now, exact solutions to the Klein–Gordon equation can be written as follows:

$$\Psi_{lm}(\mathbf{r},t)=j_l(2\beta_0\gamma R)P_l^m(\tau/R) \times\cos(m\phi)e^{-i2\beta_0\gamma\tau}e^{i\beta_0\eta}. \tag{39}$$

For azimuthally symmetric solutions ($m=0$), the zeroth order mode is given by

$$\Psi_{00}(\mathbf{r},t)=j_0(2\beta_0\gamma R)e^{-i2\beta_0\gamma\tau}e^{i\beta_0\eta}. \tag{40}$$

Its amplitude decreases as ρ^{-1} in the transverse direction and as τ^{-1} in the direction of propagation. This is a property shared by all even modes ($l=$ even integer). On the other hand, odd modes are more localized in the transverse direction. To see this, consider the first-order mode, viz.,

$$\Psi_{01}(\mathbf{r},t)=j_1(2\beta_0\gamma R)(\tau/R)e^{-i2\beta_0\gamma\tau}e^{i\beta_0\eta}. \tag{41}$$

For large arguments, $j_1(z)\simeq\sin(z-\pi/2)/z$; consequently, $\Psi_{01}(\mathbf{r},t)$ decays as ρ^{-2} in the transverse direction, but still decays as τ^{-1} in the z direction. These decay properties indicate that the solutions given in Eq. (39) have infinite total energy content, a feature they share with plane-wave solutions and Brittingham's FWMs. In analogy to the FWMs, localized slowly decaying solutions to the Klein–Gordon equation, with a finite energy content, can be synthesized as a superposition of the wave packets given in Eq. (39).

As long as $\Psi(\mathbf{r},t)$ is treated as a classical field, the kinematics of a particle represented by it can be derived from the energy and the momentum densities of a Klein–Gordon field, viz.,

$$H(\mathbf{r},t)=c^{-2}\partial_t\Psi(\mathbf{r},t)\partial_t\Psi^*(\mathbf{r},t) +\nabla\Psi(\mathbf{r},t)\cdot\nabla\Psi^*(\mathbf{r},t) +\mu^2\Psi(\mathbf{r},t)\Psi^*(\mathbf{r},t),$$

$$\mathbf{P}(\mathbf{r},t)=-c^{-2}[\partial_t\Psi(\mathbf{r},t)\nabla\Psi^*(\mathbf{r},t) +\partial_t\Psi^*(\mathbf{r},t)\nabla\Psi(\mathbf{r},t)].$$

As mentioned earlier, solutions of infinite energy content, such as those given in Eq. (39), can be superimposed to obtain finite energy ones. In this case, the integration of $H(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$ over all space will give the energy and the momentum of the particle represented by such solutions. Another possibility is to search for nondispersive bump solutions of finite energy densities. For a solution of this kind the central portion of the field has a larger energy content and small oscillations compared to the tails. The relatively large oscillations of the tails cancel out on the average when such a field interacts with a large scale measuring instrument. Space will appear to be empty except for the large amplitude, oscillation-free central portion. In this case, the energy and the momentum of the particle can be calculated by integrating the energy and momentum densities over the central part of the field. A crude example of what we mean is the integration of the one-dimensional function $\sin(x)/x$ over all values of x from $-\infty$ to $+\infty$. This will give a value of π which is approximately equal to the area under the first lobe of the function between its first two zeroes. In an interaction of such a field with another bump field (e.g., the FWM pulse), the interaction will be very large when the central parts of both fields overlap; at the same time the tails will be averaged out. In such a case the large amplitude central portions of the fields are the only parts that really contribute to the interaction and can be measured. An interaction theory is needed to provide a more rigorous and complete discussion of this possibility; the development of such a theory is out of the scope of this work.

Solutions describing nondispersive wave packets are not restricted to the form given in Eq. (39); as mentioned earlier, any solution to Eq. (36) will give a wave packet that will keep its form as it travels in free space. A special case of these solutions has been derived by MacKinnon,[7,8] who demonstrated that a de Broglie wave packet can be formed by assuming that the phase of a particle's internal vibration is independent of the choice of a reference frame. MacKinnon's solution is almost identical to the Ψ_{00} mode, especially when the terms in the exponent are rearranged so that

$$\Psi_{00}(\mathbf{r},t) = j_0(2\beta_0\gamma R)e^{-i\beta_0\gamma^2(1+v_g^2/c^2)[z-(c^2/v_g)t]}. \tag{42}$$

Because of the close resemblance of the two solutions, it is of interest to compare more closely the methods leading to them. This comparison will be carried out in the next section, where the difference between the interpretations of the solution in (39) as a classical wave function and as a quantum mechanical wave packet will be investigated. A discussion will also be provided of the dispersion relationships involved and their effect on the kinematics of a free particle represented by a wave packet such as the one in Eq. (42).

Before we proceed to the next section, it is worthwhile to point out that the de Broglie relationship between the group velocity v_g of the envelope and the phase velocity $v_{\rm ph}$ of the associated plane wave (i.e., $v_{\rm ph}v_g = c^2$) is embodied automatically in Eq. (42) by simply imposing the requirement that $\Psi(\mathbf{r},t)$ should be a nonsingular continuous wave packet that does not disperse with time. It is quite interesting that the localization requirement alone can lead to such a relationship, without any reference to an "internal clock" of the particle, or the need for the assumption that the phase of the internal clock of the particle be equal to the phase of the associated wave, two concepts utilized by de Broglie to derive the relationship $v_{\rm ph}v_g = c^2$ in his attempt to maintain the invariance of the relationship $mc^2 = h\nu$ for all frames of reference. The particle-wave velocity equation $v_{\rm ph}v_g = c^2$ has been considered[20] to be a generalization of the more limited velocity relation $v_g = v_{\rm ph} = c^2$, which is true for massless particles only. Moreover, it has been argued by MacGregor[20] that the relationship $v_{\rm ph}v_g = c^2$ should be taken as a basic postulate of special relativity, replacing the popular postulate that the speed of light in free space has the value c in all inertial frames.

IV. NONDISPERSIVE WAVE PACKETS AND DISPERSION RELATIONSHIPS

The similarity between MacKinnon's solution and the Ψ_{00} mode is very clear when we recall that $j_0(x) = \sin(x)/x$. In order to examine these two results more carefully, we first write MacKinnon's 3-D wave packet[8] as

$$\Psi_M(\mathbf{r},t) = [\sin(kR)/kR\,]e^{i[\omega(k_0)t - k_0 z]}, \tag{43}$$

where

$$k = \mu, \tag{44}$$

$$R = \sqrt{\rho^2 + \gamma^2(z-ut)^2}. \tag{45}$$

The parameter k_0 was defined by MacKinnon in the case of the 1-D solution[7] as

$$k_0 = \gamma\mu(u/c). \tag{46}$$

In the 3-D case, it is only correct up to a numerical factor of $\sqrt{2}$, as will be shown later. The frequency $\omega(k_0)$ entering into Eq. (43) was defined as

$$\omega(k_1) - \omega(k_0) = u(k_1 - k_0), \tag{47}$$

with the provision that

$$\partial_{k_0}\omega(k_0) = u \text{ and } \partial^2_{k_0}\omega(k_0) = 0. \tag{48}$$

These conditions were claimed by MacKinnon to be necessary for the wave packet to retain its form for all time. The velocity u of the particle is derived from the derivative of $\omega(k_0)$ with respect to k_0. However, the explicit dependence of $\omega(k_0)$ on k_0 is not very obvious, and the adequacy of the definition given by (47) is questionable.

Our aim in this section is to clarify these issues through a detailed analysis of the properties of the solutions given in Eqs. (42) and (43). The main difference between the two solutions is that Ψ_{00} has been treated, until now, as a classical nondispersive wave packet with an envelope that moves with a velocity v_g defined in Eq. (34). This is not, however, a unique velocity as will be shown in this section. The wave function Ψ_M, on the other hand, is considered to be a quantum mechanical entity moving with a velocity u derived from a dispersion relationship as in Eq. (48). In order to compare the two wave functions, we will consider Ψ_{00}, for the rest of this section, to be a quantum mechanical wave packet. In this case, the group velocity v_g might not be consistent with the fact that the kinematics of a particle should

be derived from its phase factor. To check such an inconsistency we can refer to the particle's energy and momentum relationships. As stated earlier, the energy and the momentum can be calculated by taking the derivatives of the phase of Ψ_{00} with respect to time and space, respectively, viz.,

$$E = \hbar\partial_t\phi, \quad p = -\hbar\nabla\phi. \tag{49}$$

Using $\Psi_{00}(\mathbf{r},t)$ in Eq. (42) together with definition of v_g given by Eq. (34), we obtain the following expressions for the energy and the z component of the momentum:

$$E = \frac{c^2}{[1-v_g^2/c^2]^{1/2}} m_0 \frac{[1+v_g^2/c^2]}{[1-v_g^2/c^2]}, \tag{50a}$$

$$P_z = \frac{v_g}{[1-v_g^2/c^2]^{1/2}} m_0 \frac{[1+v_g^2/c^2]}{[1-v_g^2/c^2]}. \tag{50b}$$

These are incorrect expressions unless we use an apparent rest mass

$$M_0 = m_0 \frac{[1+v_g^2/c^2]}{[1-v_g^2/c^2]}, \tag{51}$$

which is identical to the "apparent mass" introduced by de Broglie[9] in order to guarantee the consistency of the equations of motion of particles represented by such wave packets. The apparent mass is defined as

$$M_0 = \sqrt{m_0^2 + \delta m_0^2}, \tag{52a}$$

$$\delta m_0^2 = (\hbar^2/c^2)(1/\psi)(c^{-2}\partial_t^2 - \nabla^2)\psi. \tag{52b}$$

The quantity ψ in Eq. (52b) is defined through the relationship $\Psi(\mathbf{r},t) = \psi[R(\mathbf{r},t)]f(z,t)$. To arrive at the definition of M_0 given in (52), one should take into account that $R = \sqrt{\rho^2 + \gamma^2(z-v_g t)^2}$ and that β_0 is related to v_g through Eq. (34), from which one has $\beta_0^2 = \mu^2(v_g^2/c^2)/(1-v_g^2/c^2)$.

The redefinition of the mass M_0, as given in (51), produces the expected energy and momentum relations. The results are physically unattractive, however, because of the dependence of M_0 on v_g. On the other hand, MacKinnon[8] has indicated that his solution cannot suffer from such a problem because, for $|\psi| = \sin(\psi R)/\psi R$, it follows that $\delta m_0 = \mu\hbar/c$ and the apparent mass reduces to

$$M_0 = \sqrt{2}m_0. \tag{53}$$

To overcome the difficulty associated with the solution $\Psi_{00}(\mathbf{r},t)$ obtained by utilizing the bidirectional representation, we can start with the ansatz

$$\Psi(\mathbf{r},t) = G(\boldsymbol{\rho},z,t)e^{i\beta_0(z+(c^2/u)t)}, \tag{54}$$

where, now, the particle velocity, designated by u, is left undefined. Substitution of (54) into Eq. (20) gives a generalization of the partial differential Eq. (33), viz.,

$$i2\beta_0(\partial_z - u^{-1}\partial_t)G(\boldsymbol{\rho},z,t) + (\partial_z^2 - c^{-2}\partial_t^2)G(\boldsymbol{\rho},z,t) + \nabla_T^2 G(\boldsymbol{\rho},z,t) + (\beta_0^2\gamma^{-2}(c^2/u^2) - \mu^2)G(\boldsymbol{\rho},z,t) = 0, \tag{55}$$

where

$$\gamma = (1-u^2/c^2)^{-1/2}.$$

Motivated by the convection term on the left-hand side of Eq. (55), we can choose

$$G(\boldsymbol{\rho},z,t) = g(\boldsymbol{\rho},\tau)e^{-i2\beta_0\gamma\tau}, \quad \tau = \gamma(z-ut), \tag{56}$$

which reduces (55) into a Helmholtz equation; specifically,

$$\nabla_T^2 g(\boldsymbol{\rho},\tau) + \partial_\tau^2 g(\boldsymbol{\rho},\tau) + \chi^2 g(\boldsymbol{\rho},\tau) = 0, \tag{57a}$$

$$\chi^2 = 4\beta_0^2\gamma^2 + (\beta_0^2/\gamma^2)(c^2/u^2) - \mu^2. \tag{57b}$$

A solution to the Klein–Gordon equation can be written now as follows:

$$\Psi(\mathbf{r},t) = [\sin(\chi R)/\chi R] \times \exp[i\beta_0(1+u^2/c^2)\gamma^2(z-(c^2/u)t)]. \tag{58}$$

It should be noted that χ is identical to MacKinnon's k. The value of $u = u(\beta_0)$ can be deduced from the algebraic relationship (57b). Instead, we introduce the change of variables

$$k_0 = \beta_0\gamma^2(1+u^2/c^2), \tag{59}$$

which yields, upon substitution into Eq. (57b), the following expression for the velocity:

$$u = \pm ck_0/\sqrt{k_0^2 + \chi^2 + \mu^2}. \tag{60}$$

In the case of MacKinnon's wave packet, u was treated as a parameter independent of k_0. However, such an assumption does not make sense because the relationship $p_z = \hbar k_0$ for the momentum implies that p_z depends on k_0, and one expects the velocity to change as the momentum varies.

The definition of k_0 given in (59) changes the wave packet into the form

$$\Psi(\mathbf{r},t) = [\sin(\chi R)/\chi R]e^{i[\omega(k_0)t - k_0 z]}, \tag{61}$$

where

$$\omega(k_0) = (c^2/u)k_0. \tag{62}$$

An explicit dispersion relationship for $\omega(k_0)$ can be found by combining Eqs. (60) and (62); specifically,

$$\omega(k_0) = \pm c\sqrt{k_0^2 + \chi^2 + \mu^2}. \tag{63}$$

The positive and negative signs correspond to positive and negative energies, respectively. It is tempting to think of χ and k_0 as transverse and longitudinal wave numbers, respectively. This is not the case, however, and for the wave packet to represent a quantum mechanical particle moving in free space, we need to introduce the notion of an apparent mass M_0, as defined in Eq. (52). It is straightforward to show that $M_0 = (\chi^2 + \mu^2)^{1/2}\hbar/c$, and using Eq. (63) we arrive at the familiar energy momentum relationship

$$E = \pm c\sqrt{p^2 + M_0^2 c^2}, \tag{64}$$

where we have made use of the relationships $p = \hbar k_0$ and $E = \hbar\omega(k_0)$.

If we choose $\chi = \mu$, the velocity relationship reduces to

$$u = ck_0/\sqrt{k_0^2 + 2\mu^2}, \tag{65}$$

which resembles the group velocity of a 1-D wave packet with an apparent mass $M_0 = \sqrt{2}\mu\hbar/c$. Furthermore, using Eq. (65), an expression for k_0 can be easily derived, viz.,

$$k_0 = \sqrt{2}\gamma\mu(u/c), \tag{66}$$

which is the correct definition of k_0 for the 3-D wave packet.

It should be pointed out that the velocity expression given in Eq. (60) satisfies neither (47) nor the second provision in Eq. (48), i.e., the conditions claimed by MacKinnon as necessary for the construction of nondispersive wave packets. The similarity between the definitions of u and v_g should, also, be noted. Beside the factor of $\sqrt{2}$, which appears in the apparent mass, i.e., $\mu^2 \to 2\mu^2 = (M_0/\hbar c)^2$, the main difference between the expressions (34) and (65) is that β_0 is replaced by k_0. The velocity u, on the other hand, leads to the correct kinematics only because the momentum and energy operators are specified as in Eq. (49). If these operators are defined differently, we need a velocity different from u to get the correct kinematics.

It can be deduced from the comparison carried out in this section that the velocity u and the wave numbers β_0 or k_0 enter as parameters that can be defined freely within the limits set up by the dispersion condition (57b). The transformation (59) was introduced in order to demonstrate that one can arrive at MacKinnon's solution as a special case for the choice of u and β_0. It is very important to emphasize this freedom and to point out that different choices can lead to various kinematics depending on the manner in which the quantum mechanical operators are specified.

V. THE DIRAC EQUATION

The exposition given in Sec. II might give one the impression that the bidirectional representation is only applicable to second-order equations that are quadratic in the time derivative. This is not the case, since it can be applied to the Schrödinger equation as well as the Dirac equation. In this section, the de Broglie wave packet derived in the case of a scalar Klein–Gordon field will be used to find nondispersive wave packets for the vector fields representing massive spin 1/2 fermions. Such particles are naturally represented by the Dirac equation. It is well known, however, that fermions can be represented rather satisfactorily by a spinorial form of the Klein–Gordon equation.[21]

We begin with the second-order equation:

$$(ic^{-1}\partial_t + \boldsymbol{\sigma}\cdot\nabla)(ic^{-1}\partial_t - \boldsymbol{\sigma}\cdot\nabla)\phi(\mathbf{r},t) = \mu^2\phi(\mathbf{r},t), \tag{67}$$

where $\boldsymbol{\sigma}$ are Pauli matrices, viz.,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{68}$$

and $\phi(\mathbf{r},t)$ is a two-component spinor. Making use of the properties of the Pauli matrices it can be shown that Eq. (67) can be reduced to the two-component spinorial Klein–Gordon equation:

$$(c^{-2}\partial_t^2 - \nabla^2)\phi(\mathbf{r},t) + \mu^2\phi(\mathbf{r},t) = 0. \tag{69}$$

To find a nondispersive packet solution representing a massive spin 1/2 field, we can choose a solution to Eq. (69) similar to that given in Eq. (61); namely,

$$\phi(\mathbf{r},t) = \begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix} j_0(\chi R) e^{i(\omega(k_0)t - k_0 z)}. \tag{70}$$

This spinor field can be used to derive solutions to the Dirac equation:

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \mu\right)\Psi_D(\mathbf{r},t) = 0, \tag{71}$$

The definitions of the gamma matrices entering into this equation are given in Ref. 21; $\Psi_D(\mathbf{r},t)$ is a four-component spinor defined as follows:

$$\Psi_D(\mathbf{r},t) = \begin{bmatrix} \phi^R(\mathbf{r},t) + \phi^L(\mathbf{r},t) \\ \phi^R(\mathbf{r},t) - \phi^L(\mathbf{r},t) \end{bmatrix}. \tag{72}$$

The two-component spinors $\phi^L(\mathbf{r},t)$ and $\phi^R(\mathbf{r},t)$ are related to $\phi(\mathbf{r},t)$ given in Eq. (70) as follows:

$$\phi^L(\mathbf{r},t) = \phi(\mathbf{r},t), \tag{73a}$$

$$\phi^R(\mathbf{r},t) = (i/\mu)(c^{-1}\partial_t - \boldsymbol{\sigma}\cdot\nabla)\phi(\mathbf{r},t). \tag{73b}$$

Carrying out the operations indicated in (73b), we find that

$$\Psi_D(\mathbf{r},t) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} e^{i(\omega t - k_0 z)}, \tag{74}$$

where

$$\psi_1 = \phi_a\left\{i\left(1 + \frac{u}{c}\right)\chi j_1(\chi R)\gamma^2\frac{(z-ut)}{\mu R} + \left(1 - \frac{\omega}{\mu c} - \frac{k_0}{\mu}\right)j_0(\chi R)\right\} + i\phi_b\chi j_1(\chi R)\frac{(x-iy)}{\mu R}, \tag{74a}$$

$$\psi_2 = i\phi_a\chi j_1(\chi R)\frac{(x+iy)}{\mu R} - \phi_b\left\{i\left(1 - \frac{u}{c}\right)\chi \times j_1(\chi R)\gamma^2\frac{(z-ut)}{\mu R} - \left(1 - \frac{\omega}{\mu c} + \frac{\kappa_0}{\mu}\right)j_0(\chi R)\right\}, \tag{74b}$$

$$\psi_3 = \phi_a\left\{i\left(1 + \frac{u}{c}\right)\chi j_1(\chi R)\gamma^2\frac{(z-ut)}{\mu R} \times -\left(1 + \frac{\omega}{\mu c} + \frac{k_0}{\mu}\right)j_0(\chi R)\right\} + i\phi_b\chi j_1(\chi R)\frac{(x-iy)}{\mu R}, \tag{74c}$$

$$\psi_4 = i\phi_a\chi j_1(\chi R)\frac{(x+iy)}{\mu R} - \phi_b\left\{i\left(1 - \frac{u}{c}\right)\chi \times j_1(\chi R)\gamma^2\frac{(z-ut)}{\mu R} + \left(1 + \frac{\omega}{\mu c} - \frac{k_0}{\mu}\right)j_0(\chi R)\right\}. \tag{74d}$$

The four independent solutions to the Dirac equation can be directly obtained from Eq. (74) using the negative and positive energy values of $\omega(k_0)$, in addition to choosing ϕ_a and ϕ_b so that two independent solutions for $\phi(\mathbf{r},t)$ can be obtained, e.g., $\phi_a = 0, \phi_b = 1$ and $\phi_a = 1, \phi_b = 0$. These solutions seem to be quite complicated; nevertheless, they represent a field peaked around the origin that travels in a straight line in free space and does not disperse for all time. Despite the complicated form of the solutions, still some physical results can be obtained. For example, the four independent solutions given in Eqs. (74) are not eigenspinors of the helicity operator Σ_z defined as

$$\Sigma_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}. \tag{75}$$

Moreover, if we choose $\phi_a = 1$ and $\phi_b = 0$, the solution given in (74) is still not an eigenstate. We are mainly interested, however, in the large amplitude portion of the field around the center of the pulse ($x = 0, y = 0, z = ut$). In this portion, $j_1(\chi R) \simeq 0$, while $j_0(\chi R) \simeq 1$. Therefore, the components of the spinor given in Eq. (74) can be approximated around the center of the pulse as

$$\psi_1 \simeq 1 - \omega/\mu c - k_0/\mu, \quad \psi_2 \simeq 0,$$
$$\psi_3 \simeq 1 + \omega/\mu c + k_0/\mu, \quad \psi_4 \simeq 0,$$

and $\Psi_D(\mathbf{r},t)$ becomes an eigenspinor of the helicity operator with an eigenvalue $+1$. The same argument can be repeated for $\phi_a = 0$ and $\phi_b = 1$ in order to obtain an eigenspinor with an eigenvalue equal to -1. Similarly, we can get two independent eigenspinors for negative energies with eigenvalues $+1$ and -1.

VI. CONCLUSIONS

The bidirectional representation has been used to derive localized, nondispersive solutions to the Klein–Gordon equation by reducing it to a Helmholtz equation with its z coordinate replaced by the translational variable $\tau = \gamma(z - ut)$. The ansatz leading to such a reduction allows one to derive systematically a large class of nondispersive wave packets, representing massive particles, by making use of the known solutions to the Helmholtz equation. In seeking solutions of this type the particle-wave velocity relationship $v_{\rm ph} v_g = c^2$ follows automatically from the sole requirement of particle localization. The importance of this result need not be emphasized. It is quite intriguing, however, that in order to derive a nondispersive localized solution to the Klein–Gordon equation we arrive at a relationship that guarantees the Lorentz invariance of the formula $h\nu = mc^2$ and which can be used to generalize the postulates of spectial relativity.[20]

A special case of the solutions derived in connection with the Klein–Gordon equation was MacKinnon's nondispersive wave packet. A comparison of this packet to our results helped in clarifying some of the subtleties in MacKinnon's solution; his parameters k, k_0 are now well defined and an explicit form of the dispersion relationship $\omega(k_0)$ has been derived. The derivative of $\omega(k_0)$ with respect to k_0 gives an expression of the velocity which does not satisfy Eq. (47); furthermore, $\omega(k_0)$ is a nonlinear function of k_0, thus violating MacKinnon's condition $\partial^2_{k_0}\, \omega(k_0) = 0$. The dependence of the velocity on k_0 is expected if one recalls the momentum relationship $p_z = \hbar k_0$; as the momentum of the particle increases, one expects the group velocity of the wave packet representing the particle to increase also.

It has been shown that the apparent mass introduced by de Broglie has to be used in order to obtain the correct energy and momentum describing the motion of massive particles. For the specific wave packet given in Eq. (58) the apparent mass has the value $\hbar c^{-1}(\chi^2 + \mu^2)^{1/2}$. Choosing χ to be proportional to μ through a numerical factor independent of v_g, it follows that M_0 is proportional to the rest mass m_0. On the other hand, if χ is chosen to depend on v_g, the apparent mass M_0 depends on the velocity of the particle, a property which is not very attractive.

The results obtained for the case of the scalar massive Klein–Gordon fields were extended to the spinor massive fields governed by the Dirac equation giving de Broglie nondispersive wave packets representing free massive fermions. This particular application demonstrates that bidirectional solutions can also be derived for field equations characterized by first-order time derivatives. Similar solutions can be obtained for the Schrödinger equation; however, we prefer to publish these results separately because of their relevance to an interesting class of nondispersive solutions to the Schrödinger equation introduced by Berry and Balazs.[22]

In summary, localized, nonsingular, and nondispersive solutions have been derived to linear equations governing the motion of massive particles; specifically, the Klein–Gordon equation and the Dirac equation. Unlike soliton solutions to nonlinear equations, these are solutions to linear equations that can explain the localization properties of particles, at least in free motion. If $\Psi(\mathbf{r},t)$ is treated as a quantum mechanical wave packet, the kinematics of a particle represented by such a field are derived from its phase. On the other hand, if we consider $\Psi(\mathbf{r},t)$ to be a classical field, the kinematics are derived from the energy and momentum densities. A linear superposition can be used to construct finite energy, slowly spreading wave packets. As a consequence, an integration over all space of the field's energy and momentum densities will give the particle's energy and momentum. Another possibility is to derive nonsingular bump field solutions (not neccessarily of finite total energy content) of a large amplitude at the center and much smaller amplitudes but high oscillations at the tails. During an interaction these tails are averaged out and only the central portion of the field can be felt. The kinematics of a particle are, thus, related to the momentum and energy content of the central field. Such localized bump solutions are incorporated in an extended wave field. Using this property, we have been able to justify the wave-particle dualism.[23] We have also been able to provide a novel interpretation of Young's double slit experiment.[24]

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[1] J. N. Brittingham, J. Appl. Phys. **54**, 1179 (1983).
[2] A. Sezginer, J. Appl. Phys. **57**, 678 (1985).
[3] P. A. Belanger, J. Opt. Soc. Am. A **1**(7), 723 (1984).
[4] R. W. Ziolkowski, J. Math. Phys. **26**, 861 (1985).
[5] L. de Broglie, *The Current Interpretation of Wave Mechanics* (Elsevier, Amsterdam, 1964).
[6] G. Lochak, in *The Wave-Particle Dualism*, edited by S. Diner, D. Fargue, G. Lochak, and F. Selleri (Reidel, Dordrecht, 1984).
[7] L. MacKinnon, Found. Phys. **8**, 157 (1978).
[8] L. MacKinnon, Lett. Nuovo Cimento **31**, 37 (1981).
[9] L. de Broglie, *Nonlinear Wave Mechanics* (Elsevier, Amsterdam, 1960).
[10] E. Madelung, Z. Phys. **40**, 322 (1926).
[11] D. Bohm, Phys. Rev. **85**, 166 (1952).
[12] D. Bohm, Phys. Rev. **85**, 180 (1952).
[13] D. Fargue, in *The Wave-Particle Dualism*, edited by S. Diner, D. Fargue, G. Lochak, and F. Selleri (Reidel, Dordrecht, 1984).
[14] R. W. Ziolkowski, A. M. Shaarawi, and I. M. Besieris, Nucl. Phys. B (Proceedings supplement) **6**, 255 (1989).
[15] P. Hillion, J. Math. Phys. **28**, 1743 (1987).
[16] P. Hillion, Int. J. Theor. Phys. **27**, 57 (1988).
[17] I. M. Besieris, A. M. Shaarawi, and R. W. Ziolkowski, J. Math. Phys. **30**, 1254 (1989).
[18] R. W. Ziolkowski, D. K. Lewis, and B. D. Cook, Phys. Rev. Lett. **62**, 147 (1989).
[19] R. W. Ziolkowski, Phys. Rev. A **39**, 2005 (1989).
[20] M. H. MacGregor, Lett. Nuovo Cimento **43**, 49 (1985).
[21] J. J. Sakurai, *Advanced Quantum Mechanics* (Benjamin/Cummings, Menlo Park, 1984).
[22] M. V. Berry and N. L. Balazs, Am. J. Phys. **47**, 264 (1979).
[23] A. M. Shaarawi, I. M. Besieris, and R. W. Ziolkowski, submitted to found. of Phys.
[24] A. M. Shaarawi, I. M. Besieris, and R. W. Ziolkowski, submitted to Phys. Rev. A.