Solutions of nonlinear partial differential equations in phase space

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Abstract

In this paper we consider the problem of constructing solutions of several well known nonlinear partial differential equations (p.d.e.s) in phase space (i.e. the Fourier transform domain). We seek solutions representing travelling focussed pulses. As such, based on a technique used to construct such solutions (so called *Localized Wave* solutions) of linear p.d.e.s, we look for phase space solutions consisting of a generalized function whose support is a particular line or surface, together with a suitable weighting function. The support of the phase space solution must be such that it regenerates itself after the appropriate nonlinear operation. In one spatial dimension we construct the usual well known soliton solutions of several equations. For the case of higher spatial dimensions we construct a travelling "slab" pulse solution of the nonlinear Schrödinger equation. We also discuss some issues involved with the extra freedom one has for the phase space support, leading perhaps to more exotic spacetime domain solutions.

1. Introduction

There are various methods available for obtaining solutions of nonlinear partial differential equations (p.d.e.s) [1,2]. Of particular interest in physics and engineering are those equations having time and one or more spatial directions as variables, and for such equations one is often interested in solutions representing focussed pulses of energy. The soliton solutions of many well known p.d.e.s are examples.

Fourier transform methods traditionally are used only when dealing with linear p.d.e.s. Applying these methods to the homogeneous wave equation, for example, we have shown [3] how one can design, in phase space [i.e. the Fourier transform domain], focussed moving pulse solutions, and how one should vary the phase space characteristics of the solution in order to achieve more focussing in spacetime. Basically, the phase space solution consists of any generalized function whose support lies on a particular surface (the so-called "dispersion surface", determined by the transform of the wave operator), together with a particular weighting function. It is the behaviour

of the weighting function which determines the spacetime focussing properties of the solution. It is important to realise that the wave equation itself enters into the discussion only inasmuch as it determines the surface on which the support of the phase space solution must lie. As such, one may use transform methods to construct solutions of nonlinear p.d.e.s. In these cases, however, there will be no "dispersion surface" (in the above sense) associated with phase space solutions. One may still proceed by appealing to support arguments. That is, the effects of the various derivatives in phase space do not change the support of the prospective solution; the effects of the nonlinearity do change the support, in general, and so one must now choose as solutions those phase space functions whose supports are "regenerating" in the sense of surviving the nonlinear phase space operations.

These ideas are made more concrete below. In Section 2 we apply the method to the important nonlinear [cubic] Schrödinger equation, in one spatial dimension, and show how the usual soliton solution may be constructed. We discuss examples of other nonlinear p.d.e.s to which we have applied this method. In Section 3 we consider the nonlinear Schrödinger equation in three spatial dimensions and demonstrate how the above method can be used to construct "slab-like" moving solutions. The introduction of more spatial dimensions implies potentially more freedom in phase space when constructing supports having the requisite "regenerating" properties. We discuss this freedom, and suggest an example of a tentative support having the necessary "regenerating" properties. In the final section we summarise the approach.

2. One spatial dimension

For exemplification of the method we shall start by considering the [cubic] nonlinear Schrödinger equation in one spatial and one temporal dimension:

$$i\frac{\partial}{\partial t}u(z,t) + \frac{\partial^2}{\partial z^2}u(z,t) + \alpha_0|u(z,t)|^2u(z,t) = 0, \tag{1}$$

where $\alpha_0 > 0$ is a constant. If we apply a spacetime Fourier transform to (1) we obtain

$$\omega U(k_z,\omega) - k_z^2 U(k_z,\omega) + \frac{\alpha_0}{(2\pi)^4}[U(k_z,\omega) *_{k_z,\omega} U^*(-k_z,-\omega)] *_{k_z,\omega} U(k_z,\omega) = 0, \tag{2}$$

where $*_{k_z,\omega}$ denotes the operation of convolution in the variables k_z and ω, and where

$$U(k_z,\omega) \equiv \mathcal{F}_{z,t}\{u\}(k_z,\omega) \equiv \mathcal{F}_z\mathcal{F}_t\{u\}(k_z,\omega)$$

$$\equiv \int_{\mathbb{R}} dz \int_{\mathbb{R}} dt\, e^{-ik_z z}\, e^{i\omega t}\, u(z,t), \tag{3}$$

and so $u(z,t) \leftrightarrow U(k_z,\omega)$ denotes a Fourier transform pair. Note that we have the transform pair

$$|u(z,t)|^2 \leftrightarrow \frac{1}{(2\pi)^2}U(k_z,\omega) *_{k_z,\omega} U^*(-k_z,-\omega), \tag{4}$$

which may perhaps best be thought of as a type of "autocorrelation" of U with itself. Nevertheless, for ease of referral we shall henceforth refer to the spacetime Fourier transform of $|u(z,t)|^2u(z,t)$ as involving a double convolution of $U(k_z,\omega)$ with itself, despite the special meaning of one of the convolutions.
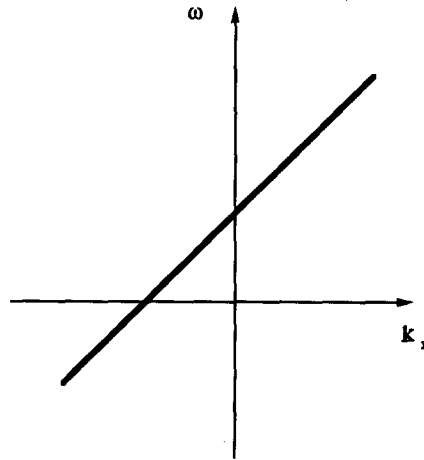
Fig. 1. Tentative choice of "regenerating" support line ($\omega = mk_z + b$) for phase space solution of nonlinear Schrödinger equation.

From a knowledge of the design of *Localized Wave* (i.e. focussed) pulse solutions of the homogeneous wave equation [3] we realise that focussed, moving spacetime pulses can be associated with phase space (i.e. Fourier transform domain) functions consisting of a generalized function whose support lies on a line or surface in phase space, together with an appropriate weighting function. The decay of the weighting function for large values of the spatial transform variable(s) determines the degree of focussing of the pulse. This is independent of the particular spacetime equation (or phase space equation) to be satisfied by the pulse. In this case, however, we seek such phase space forms that satisfy (2).

If we want $U(k_z, \omega)$ to be a generalized function with a particular support in phase space, together with an associated weighting function, then with regard to (2) we realise that multiplying $U(k_z, \omega)$ by either ω or k_z^2 will not change the support. However, in general the support of even $U(k_z, \omega) *_{k_z, \omega} U^*(-k_z, -\omega)$ will differ from that of $U(k_z, \omega)$, as will indeed that of the double convolution term in (2). Thus, we must seek solutions of (2) in the desired form (i.e. product of a generalized function with a particular support, together with an associated weighting) such that the support of $U(k_z, \omega)$ is "regenerating" after the double convolution, in the sense that it equals the support of $[U(k_z, \omega) * U^*(-k_z, -\omega)] * U(k_z, \omega)$.

One obvious choice of a"regenerating" support is the arbitrary straight line, of slope m and ω-axis intercept b, as shown in Fig. 1. With the support given in Fig. 1, we see that $U(k_z, \omega)$ must be given by

$$U(k_z, \omega) = F(\omega) \delta(\omega - mk_z - b), \qquad (5)$$

where $F(\omega)$ is, as yet, an arbitrary weighting function.

The statement that the support of $U(k_z, \omega)$ regenerates itself after the double convolution in (1) is perhaps best justified graphically; this facility, although clearly not necessary here, is useful in the multidimensional case. The first convolution operation involves multiplying $U(k_z, \omega)$ by the conjugate of linear translates of itself, and then integrating over the k_z, ω plane. The product will be nonzero (i.e. the two support lines will overlap) only where the support line is translated so that it lies on itself, so that the translates must lie on a line parallel to the original support line, passing

through the origin. Performing the first convolution gives specifically

$$U(k_z,\omega) * U^*(-k_z,-\omega) = [F(\omega) *_\omega F^*(-\omega)]\frac{\delta(\omega - mk_z)}{m},\tag{6}$$

where $*_\omega$ denotes convolution with respect to ω only, and we see that the support of this function is not in general the same as that of $U(k_z,\omega)$. However, repeating the convolution process we "flip the function in (6) over" with respect to the two axes (this does not change the support), then multiply translates of it by $U(k_z,\omega)$, and then integrate over the k_z,ω plane. The two lines will overlap only when the translates lie on the original support line of $U(k_z,\omega)$. Specifically,

$$[U(k_z,\omega) *_\omega U^*(-k_z,-\omega)] * U(k_z,\omega) = [F(\omega) * F^*(-\omega)] * F(\omega)\frac{\delta(\omega - mk_z - b)}{m^2}.\tag{7}$$

As the support of the double convolution term in (7) is the same as that of $U(k_z,\omega)$, in (5), we can substitute these expressions into (2) to obtain

$$\omega F(\omega)\,\delta(\omega - mk_z - b) - \frac{(\omega - b)^2}{m^2}F(\omega)\,\delta(\omega - mk_z - b)$$
$$+ \frac{\alpha_0}{(2\pi)^4}[F(\omega) * F^*(-\omega)] * F(w)\frac{\delta(\omega - mk_z - b)}{m^2} = 0,\tag{8}$$

where we have used the property $g(\xi)\,\delta(\xi - \xi_0) \equiv g(\xi_0)\,\delta(\xi - \xi_0)$ (effectively we have coupled k_z to ω in a linear relation by the choice of a straight line support in (5)). A solution of (8) is given by

$$\omega F(\omega) - \frac{(\omega - b)^2}{m^2}F(\omega) + \frac{\alpha_0}{m^2(2\pi)^4}[F(\omega) * F^*(-\omega)] * F(\omega) = 0,\tag{9}$$

and so the problem now becomes that of finding $F(\omega)$.

To this end we shall apply an inverse temporal Fourier transform to (9). The resulting equation will contain time derivatives of $f(t)$, where $f(t) \leftrightarrow F(\omega)$, and it transpires that the equation can be readily solved if the term containing the single derivative vanishes. From (9) this implies that the term in ω vanishes, and so we shall impose the constraint

$$\frac{2b}{m^2} = -1.\tag{10}$$

With this constraint satisfied we inverse transform (9) and rearrange to get

$$\frac{d^2 f(t)}{dt^2} = b^2 f(t) - \frac{\alpha_0}{(2\pi)^2}|f(t)|^2 f(t) = 0.\tag{11}$$

If we assume that $f(t)$ is real valued, then a solution of (11) is readily obtained:

$$f(t) = 2\pi b\sqrt{\frac{2}{\alpha_0}}\,\mathrm{sech}[b(t - t_0)],\tag{12}$$

where t_0 is constant.

Finally then, inverse Fourier transforming (5) gives

$$u(z,t) = \frac{e^{-izb/m}}{2\pi m} f(t - z/m),$$ (13)

and so

$$u(z,t) = \frac{b\,e^{-izb/m}}{m} \sqrt{\frac{2}{\alpha_0}} \operatorname{sech}\left(\frac{b}{m}[z - m(t - t_0)]\right),$$ (13')

subject to the constraint (10). This is a well known soliton solution of (1) [1,2].

Some comments are in order regarding the method just used. It was based upon the fact that the support of the proposed phase space solution, (5), was "regenerating" in that after convolving twice, the support of $[U(k_z,\omega) * U^*(-k_z,-\omega)] * U(k_z,\omega)$ was the same as that of $U(k_z,\omega)$. However, to be precise, this is not sufficient, as we could replace δ by δ' in (5) and still maintain the "regenerating" support argument. The problem would be that now after convolving twice we would obtain finally δ''' instead of δ'. We must therefore restrict ourselves to delta functions in phase space.

The slope and ω-axis intercept of the support line in (5) are arbitrary as far as the "regenerating" property goes. The constraint (10) is imposed only to obtain a temporal differential equation, (11), containing no first derivatives, which can be readily solved.

It is also important to note that this approach is not the same as assuming a solution containing a function $g(z - mt)$, and then substituting into (1), as one must assume a priori the existence of a phase term to make progress towards a solution [2, Section 17.8]. In our approach that phase term arises from the offset of the support line from the origin.

The above method can be applied readily to other nonlinear p.d.e.s, such as

$$\frac{\partial}{\partial t}v(z,t) + \alpha_0 v(z,t)\frac{\partial}{\partial z}v(z,t) + \frac{\partial^3}{\partial z^3}v(z,t) = 0,$$ (14)

$$\frac{\partial}{\partial t}v(z,t) - v(z,t)\frac{\partial}{\partial z}v(z,t) - \alpha_0 \frac{\partial}{\partial z^2}v(z,t) = 0,$$ (15)

$$\frac{\partial^2}{\partial t^2}v(z,t) - \frac{\partial^2}{\partial z^2}v(z,t) + \sin v(z,t) = 0,$$ (16)

which are the Korteweg–de Vries equation, Burger's equation, and the sine-Gordon equation, respectively. In the case of the first two equations, the $\partial v/\partial z$ carries over to $ik_z V(k_z,\omega)$ in the transform domain, which does not affect the support of $V(k_z,\omega)$. Hence, in both these equations we seek phase space solutions whose supports regenerate themselves after a single convolution; there will therefore be no phase terms present in the spacetime solutions, since there can be no offset of the support line from the origin in phase space. In the case of the sine-Gordon equation, by expanding $\sin v = v - v^3/3! + v^5/5! - \dots$ we realise that we need a phase space solution whose supports regenerate themselves after two [strict] convolutions (since they will thus regenerate themselves after four convolutions, and so on). We see here also then that there will be no phase terms present in the spacetime solution. Following the procedure outlined above we obtain the standard soliton pulse solutions (e.g. [2]) to these equations.

3. More than one spatial dimension

The above notions can be used to search for solutions of nonlinear p.d.e.s in more than one spatial dimension. As an example, consider the cubic Schrödinger equation in three spatial variables, x, y, z:

$$i\frac{\partial}{\partial t}u(r,t) + \nabla^2 u(r,t) + \alpha_0|u(r,t)|^2 u(r,t) = 0. \tag{17}$$

Applying a spacetime Fourier transform gives

$$\omega U(k,\omega) - k^2 U(k,\omega) + \frac{\alpha_0}{(2\pi)^8}[U(k,\omega) * U^*(-k,-\omega)] * U(k,\omega) = 0, \tag{18}$$

where $k^2 = |k|^2 = k_x^2 + k_y^2 + k_z^2$. We seek a solution $U(k,\omega)$ whose support is "regenerating" in the sense of surviving the double convolution. An obvious choice is therefore the product of a delta function, whose support is an arbitrary straight line in k_x, k_y, k_z, ω space, with a weighting function:

$$V(k,\omega) = F(\omega)\,\delta(\omega - m_x k_x - b_x)\,\delta(\omega - m_y k_y - b_y)\,\delta(\omega - m_z k_z - b_z). \tag{19}$$

That the support of $V(k,\omega)$ is indeed regenerating is readily verified. Substituting from (19) into (18) gives, after performing the delta function convolutions and equating the entire coefficient of the product of delta functions to zero

$$\omega F(\omega) - \left[\left(\frac{\omega - b_x}{m_x}\right)^2 + \left(\frac{\omega - b_y}{m_y}\right)^2 + \left(\frac{\omega - b_z}{m_z}\right)^2\right]F(\omega)$$

$$+ \frac{\alpha_0}{(m_x m_y m_z)^2 (2\pi)^8}[F(\omega) * F^*(-\omega)] * F(\omega) = 0, \tag{20}$$

which we must solve for $F(\omega)$. If we apply an inverse temporal transform to (20) and enforce the constraint

$$\frac{b_x}{m_x^2} + \frac{b_y}{m_y^2} + \frac{b_z}{m_z^2} = -\frac{1}{2}, \tag{21}$$

we get a time domain equation containing no first derivative terms, identical to (11) apart from constants. The solution is

$$f(t) = (2\pi)^3 m_x m_y m_z |b_m|\sqrt{\frac{2}{\alpha_0}}\mathrm{sech}\left(\frac{|b_m|}{|m|}t\right), \tag{22}$$

where $f(t) \leftrightarrow F(\omega)$, and

$$b_m = \left(\frac{b_x}{m_x}, \frac{b_y}{m_y}, \frac{b_z}{m_z}\right), \qquad m = \left(\frac{1}{m_x}, \frac{1}{m_y}, \frac{1}{m_z}\right). \tag{23}$$

Finally, then, inverse transforming (19) gives us

$$u(r,t) = \mathrm{e}^{-ir \cdot b_m}\,|b_m|\sqrt{\frac{2}{\alpha_0}}\,\mathrm{sech}\left[|b_m|\left(r \cdot \hat{m} - \frac{(t - t_0)}{|m|}\right)\right], \tag{24}$$

where $\hat{m} = m/|m|$. Note that the constraint equation (21) can be rewritten:

$$\boldsymbol{b}_m \cdot \boldsymbol{m} = -\tfrac{1}{2}. \tag{25}$$

The solution (24) represents a pulse travelling with speed $1/|\boldsymbol{m}|$ in the direction $\hat{\boldsymbol{m}}$. If the speed of the pulse increases (so that $|\boldsymbol{m}|$ decreases), we must have $|\boldsymbol{b}_m|$ increasing, and so the term $e^{-i\boldsymbol{r}\cdot\boldsymbol{b}_m}$, representing a "rippling" in the direction $\boldsymbol{b}_m$, at an obtuse angle with respect to the direction of propagation, $\boldsymbol{m}$, oscillates more rapidly.

In (19) we chose the phase space candidate solution for the Fourier transformed cubic Schrödinger equation as having the simplest support that "regenerated" itself after the double convolution in (18): i.e. a straight line support in $\boldsymbol{k}$, ω space. This was the obvious generalization of the one spatial dimension case. If we accept a phase space candidate solution containing the term $F(\omega)\delta(\omega - mk_z - b)$ (this leads to a spacetime pulse travelling with speed m in the z-direction), then there remains to choose the variation with respect to k_x and k_y. If this variation is independent of ω, then the resultant spacetime pulse will have a transverse profile that is independent of z and t. If the k_x, k_y variation is linked to ω (hence k_z, via $\delta(\omega - mk_z - b)$) then the transverse behaviour of the corresponding spacetime pulse will depend on z and t. We now explore these possibilities in more detail.

If the variation of the candidate solution, in phase space, with respect to k_x and k_y is independent of ω, then in order to proceed as in the one dimensional case (i.e. ultimately obtain a phase space equation in ω to solve for $F(\omega)$), Eq. (18) must ultimately contain only ω variations. Two terms that tend to thwart this are $\kappa^2 U(\boldsymbol{k},\omega)$ (where $\kappa^2 = k_x^2 + k_y^2$), and the double convolution of the k_x, k_y varying term (call this term, say, $G(k_x, k_y)$). To remedy this we may demand that $\kappa^2 G(k_x, k_y) = 0$ (implying that $g(x,y)$ is a solution of Laplace's equation, where $g \leftrightarrow G$), and that the double convolution $[G(k_x, k_y) * G^*(-k_x, -k_y)] * G(k_x, k_y)$ results in a term proportional to $G(k_x, k_y)$. Alternately, we may demand that the double convolution behaves as just stipulated, but now that $G(k_x, k_y)$ is such that $\kappa^2 G(k_x, k_y)$ is proportional to $G(k_x, k_y)$ (so that $g(x, y)$ satisfies a Helmholtz-type equation). Based on investigations of these possibilities, we speculate that in either case such behavior is not possible, so that no requisite $G(k_x, k_y)$ exists.

If the variation of the candidate solution, in phase space, with respect to k_x and k_y is linked to ω then an interesting possibility arises. It may be possible to choose the variation of $U(\boldsymbol{k},\omega)$ with respect to k_x, k_y, ω (as opposed to k_z and ω) as a function (call it $G(k_x, k_y, \omega)$) whose support is a surface in k_x, k_y, ω space. Note that it is essential for the argument presented in the case of one spatial dimension that the support of the delta function in k_z, ω be an infinite straight line: a finite or semi-infinite straight line support will not "regenerate" itself in the sense described above. With this we realise then that a surface support of $G(k_x, k_y, \omega)$ must extend to $\pm\infty$ in the ω variable. The weighting on such a surface in the ω direction is just $F(\omega)$, and so we are free to choose a weighting with respect to k_x and k_y. The value of k_x^2 and k_y^2 on the surface will be linked to ω, and so the term $\kappa^2 G(k_x, k_y, \omega)$ can be rewritten $H(\omega) G(k_x, k_y, \omega)$, where $H(\omega)$ depends on the surface. We thus further require the surface support of $G(k_x, k_y, \omega)$, and the k_x, k_y weighting associated with that support, to be such that

$$
\begin{aligned}
&[U(\boldsymbol{k},\omega) * U^*(-\boldsymbol{k}, -\omega)] * U(\boldsymbol{k},\omega) \\
&\equiv [F(\omega) G(k_x, k_y, \omega) *_{k_x, k_y, \omega} F^*(-\omega) G^*(-k_x, -k_y, -\omega)] *_{k_x, k_y, \omega} F(\omega) G(k_x, k_y, \omega) \\
&\quad \times \frac{\delta(\omega - mk_z - b)}{m^2} \\
&\propto \text{constant} \times [F(\omega) *_{\omega} F^*(-\omega)] * F(\omega) G(k_x, k_y, \omega) \delta(\omega - mk_z - b). \tag{26}
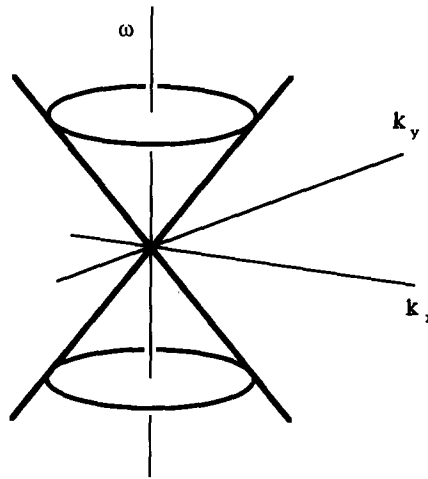\end{aligned}
$$

Fig. 2. Tentative choice of part of "regenerating" support surface for phase space solution of nonlinear Schrödinger equation in three spatial dimensions. The problem is to choose a suitable angular weighting function for the surface.

As formidable as these requirements may seem, an example of a reasonable prospective transform domain solution is the function $G(k_x, k_y, \omega)$ whose support is the cone, as shown in Fig. 2. In this case we see that

$$G(k_x, k_y, \omega) = \Phi(\varphi)\, \delta(\omega^2 - a^2 \kappa^2)\,, \tag{27}$$

where $\Phi(\varphi)$ is the weighting function on the conical surface ($\varphi = \tan^{-1}(k_y/k_x)$). That this is a reasonable guess is confirmed when we overlap the support of G with conjugated copies of itself that have been translated along the surface of the cone. In this case the two cones intersect along a generator. We see that $\kappa^2 G(k_x, k_y, \omega) = (\omega^2/a^2) G(k_x, k_y, \omega)$, and so the problem remaining is to choose the angular weighting associated with the conical surface, $\Phi(\varphi)$, such that (26) is satisfied. As yet we have been unsuccessful in this effort.

4. Concluding remarks

The method presented here, of constructing solutions of nonlinear partial diferential equations in phase space, is novel. An obvious by-product of the method is that one automatically gains knowledge of the spatial and temporal frequency content of the solution, which is of importance for many practical applications. From our knowledge of the phase space characteristics of moving, focussed, pulse-like solutions of linear p.d.e.s we realised that such phase space solutions in the nonlinear case should also be represented as generalized functions whose support is a particular line or surface, together with an appropriate weighting function associated with the surface. The support surface determines the propagation characteristics of the spacetime pulse, while the associated weighting function determines its focussing characteristics. The problem then becomes one of choosing the support surface, together with the weighting function, such that the support of the solution is "regenerating" in the sense of surviving the nonlinear [convolution] operations. In one spatial dimension the obvious choice of "regenerating" support is a straight line. In the case of higher spatial dimensions a straight line support also suffices, but there arises the possibility of more

exotic support surfaces, thus leading to interesting new spacetime solutions. Although we have not yet found examples of new support surfaces, we suggested a tentative example [for the nonlinear Schrödinger equation], in the hopes that this may spur other workers to solve this problem.

Acknowledgments

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