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Pseudodecoupling ansatz for electromagnetic aperture coupling in three dimensions

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Hertzian potentials are commonly used to describe the scattering of electromagnetic waves in three dimensions. However, when apertures are present in the scattering object, mixed boundary conditions arise and the TE and TM modes of the scattered field become coupled. Numerous attempts to decouple these TE and TM modes have led to solutions of Maxwell's equations which fail to satisfy Meixner's edge conditions. It has been found that by introducing additional gauge potentials which are homogeneous solutions of the boundary condition equations, a pseudodecoupling of the TE and TM modes can be achieved while still satisfying Meixner's edge conditions.

INTRODUCTION

The number of electromagnetic boundary value problems that can be solved exactly in three dimensions is rather small. This is especially true for scattering objects having apertures. The desire and the need for solutions to these canonical problems, however, are very strong from both theoretical and practical points of view.

In free space the electromagnetic field is derivable from two scalar functions Ψ and Φ and some specified direction in the form of Hertzian potentials [*Nisbet*, 1955]. This potential formulation has led to essentially analytical solutions of two canonical problems which bound the possible scattering geometries involving apertures. For extended bodies, the general solution for plane wave scattering from a circular hole in an infinite, perfectly conducting ground plane was given by *Meixner* [1948] and improved versions were reported by *Meixner and Andrejewski* [1950] and by *Nomura and Katsura* [1955]. For finite objects, the general solution to the problem of plane wave scattering from a circular aperture in a perfectly conducting spherical shell was constructed recently by *Ziolkowski and Johnson* [1987]. Incorrect solutions to the former problem [*Möglich*, 1927; *Meixner*, 1946] and to the normal incidence version of the latter [*Radin and Shestopalov*, 1974; *Vinogradov et al.*, 1981; *Casey*, 1981] had appeared prior to these results. The common flaw in all of those earlier works was the assumption that the treatments of Φ and Ψ could be decoupled as is possible for the corresponding closed scattering object problems. As a result, their solutions do not satisfy Meixner's edge conditions [*Jones*, 1964]. On the other hand, the common thread in *Meixner* [1948], *Meixner and Andrejewski* [1950], *Nomura and Katsura* [1955], and *Ziolkowski and Johnson* [1987] that leads to correct solutions is the recognition that a pseudodecoupling of the potentials Φ and Ψ is possible with the introduction of additional gauge potentials which, in essence, account for the presence of the aperture. These gauge potentials provide the degrees of freedom needed to construct fields that satisfy Meixner's edge conditions. Although an incorrect solution was obtained in [*Vinogradov and Shestopalov*, 1977] an analogous type of decoupling was also utilized by Vinogradov and Shestopalov for the normal incidence case of the open spherical shell problem. We speculate that this pseudodecoupling principle, which will be described in detail below, is a basic one and could lead to the solutions of several other electromagnetic canonical scattering problems involving apertures.

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DEBYE POTENTIALS

When the Hertz vectors are assumed to be the two radial vectors $\Phi\mathbf{r}$ and $\Psi\mathbf{r}$, where $\mathbf{r} = r\hat{r}$, the functions

Φ and Ψ are known as Debye potentials. In any free-space domain endowed with spherical coordinates (r, θ, ϕ) they have, depending on the desired polarization, the modal expansion forms

$$\Phi(r, \theta, \phi) = \sum_{m=0}^{\infty} \Phi_m(r, \theta) \begin{Bmatrix} -\sin m\phi \\ \cos m\phi \end{Bmatrix} \tag{1}$$

$$\Psi(r, \theta, \phi) = Y_0 \sum_{m=0}^{\infty} \Psi_m(r, \theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \tag{2}$$

where the azimuthal components

$$\begin{pmatrix} \Phi_m \\ \Psi_m \end{pmatrix}(r, \theta) = \sum_{n=m}^{\infty} \begin{pmatrix} a_{mn} \\ b_{mn} \end{pmatrix} z_n(kr) P_n^{-m}(\cos\theta) \tag{3}$$

The terms P_n^{-m} are the associated Legendre functions of degree n and order $-m$. We have chosen this convention rather than the proportional, standard choice of positive order, P_n^{+m}, because it is compatible with the nonvanishing, singular functions P_n^{-m}, $0 \le n < m$, that we will introduce below. In contrast, $P_n^{+m} \equiv 0$ for $0 \le n < m$. These expansions anticipate, of course, that we will be dealing with circular apertures in the following problems which are centered about the $\theta = 0$ axis. If, for instance, we are interested in scattering from an open spherical shell of radius a that is either empty or encloses a smaller concentric interior spherical body, the radial function $z_n(kr)$ represents a spherical Hankel function of the first kind $h_n(kr)$ of order n for $r > a$ and either a spherical Bessel function $j_n(kr)$ or linear combinations of $j_n(kr)$ and $h_n(kr)$ for $r < a$. An $e^{-i\omega t}$ time dependence is assumed and suppressed throughout. The free-space admittance $Y_0 = (\varepsilon/\mu)^{1/2}$.

Debye potentials satisfy Helmholtz's equation

$$\{\Delta + k^2\}\Phi = \{\Delta + k^2\}\Psi = 0$$

and generate the electromagnetic fields

$$\mathbf{E} = -\mathrm{curl}(\Phi\mathbf{r}) - (i\omega\varepsilon)^{-1}\mathrm{curl}\ \mathrm{curl}(\Psi\mathbf{r})$$

$$\mathbf{H} = +\mathrm{curl}(\Psi\mathbf{r}) - (i\omega\mu)^{-1}\mathrm{curl}\ \mathrm{curl}(\Phi\mathbf{r})$$

The potential Φ represents the modes TE with respect to $r(\mathrm{TE}_r)$; Ψ represents the modes TM with respect to $r(\mathrm{TM}_r)$. For scattering problems the potentials are decomposed into incident and scattered components in the interior and the exterior of the scattering object:

$$\Phi = \Phi_j^{\mathrm{inc}} + \Phi_j^s$$

$$\Psi = \Psi_j^{\mathrm{inc}} + \Psi_j^s$$

where $j = 1$ for the interior domain; $j = 2$ for the exterior. The incident field is included in both regions for convenience which means $a_{1,mn}^{\mathrm{inc}} = a_{2,mn}^{\mathrm{inc}}$ and $b_{1,mn}^{\mathrm{inc}} = b_{2,mn}^{\mathrm{inc}}$. The scattered field coefficients $a_{j,mn}^s$ and $b_{j,mn}^s$ are chosen to contain the desired solution coefficients A_{mn} and B_{mn} and appropriate terms that make E_{tan} continuous across the boundary. For example, when describing the scattering of a plane wave from an empty open spherical shell of radius a, the choice of

$$\begin{pmatrix} a_{1,mn}^s \\ b_{1,mn}^s \end{pmatrix} z_n(kr) = \begin{pmatrix} A_{mn} h_n(ka) \\ B_{mn}\, \partial_{ka}[ka h_n(ka)] \end{pmatrix} j_n(kr) \qquad r < a$$

$$\begin{pmatrix} a_{2,mn}^s \\ b_{2,mn}^s \end{pmatrix} z_n(kr) = \begin{pmatrix} A_{mn} j_n(ka) \\ B_{mn}\, \partial_{ka}[ka j_n(ka)] \end{pmatrix} h_n(kr) \qquad r > a$$

makes E_{tan} continuous across the boundary $r = a$.

OPEN SPHERE PROBLEM

First consider plane wave scattering from a closed sphere of radius a. The electromagnetic boundary conditions $E_{tan}(r = a, \theta, \phi) = 0$ require

$$\begin{aligned} &\{\sin\theta\,\partial_\theta[\partial_r(r\Psi_m)] - m(ikr\Phi_m)\}(r = a, \theta) = 0 \\ &\{m[\partial_r(r\Psi_m)] - \sin\theta\,\partial_\theta(ikr\Phi_m)\}(r = a, \theta) = 0 \end{aligned} \tag{4}$$

Since these equations are defined over the entire θ interval $[0, \pi]$, orthogonality of the functions P_n^{-m} and $\partial_\theta P_n^{-m}$ allows separation of the TE_r and TM_r modal treatments. In particular, (4) is satisfied if the azimuthal components of the Debye potentials satisfy the boundary conditions:

$$\begin{aligned} &(r\Phi_m)(r = a, \theta) = 0 \\ &[\partial_r(r\Psi_m)](r = a, \theta) = 0 \end{aligned} \tag{5}$$

On the other hand, if there is a circular hole present in the spherical shell, (4) holds only over the metal $\{(r, \theta, \phi) \mid r = a, \theta \in [0, \theta_0), \phi \in [0, 2\pi]\}$, and must be supplemented by the additional boundary conditions H_{tan} continuous over the aperture $\{(r, \theta, \phi) \mid r = a, \theta \in (\theta_0, \pi], \phi \in [0, 2\pi]\}$, which requires

$$\begin{aligned} &\{\sin\theta\,\partial_\theta[\partial_r(r\Phi_m)] - m(ikr\Psi_m)\}_{r=a-}^{r=a+} = 0 \\ &\{m[\partial_r(r\Phi_m)] - \sin\theta\,\partial_\theta(ikr\Psi_m)\}_{r=a-}^{r=a+} = 0 \end{aligned} \tag{4'}$$

The completely decoupled equations in (5) no longer specify the solution properly because the TE_r and TM_r modes are now coupled by the hole. We are then faced with a complex mixed boundary value problem defined by the coupled pair of dual series equations arising from the open sphere electro-

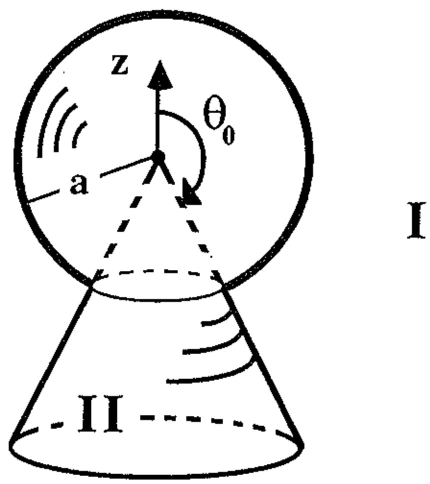

Fig. 1. The gauge potentials are defined over two regions. Region II is the cone whose apex coincides with the origin and whose intersection with the sphere $r = a$ is the rim of the circular aperture of the open spherical shell. Region I is the complement of Region II.

magnetic boundary conditions: (4) over the metal and (4′) over the aperture.

This difficulty can be circumvented by introducing additional potentials Φ^G and Ψ^G (with azimuthal expansions of the form (1) and (2), Φ_m and Ψ_m being replaced respectively with Φ_m^G and Ψ_m^G) that are homogeneous solutions to the open sphere boundary condition equations (4) and (4′). These potentials account for the presence of the hole and enable one to construct fields that satisfy Meixner's edge conditions. Their introduction also allows us to maintain the conventional TE_r and TM_r representations and permits a separate treatment of the TE_r and TM_r potentials in analogy with the closed sphere problem. However, in contrast to that analysis, the open sphere potentials remain coupled by the expansion coefficients of Φ^G and Ψ^G. Thus we achieve only a "pseudodecoupling" of the dual series equations for the TE_r and TM_r potentials. Failure to account for this coupling results in currents and fields that do not have the correct singular behavior near the edge of the aperture.

Homogeneous solutions of the open sphere boundary condition equations are readily constructed. They are used to define the auxiliary potentials Φ^G and Ψ^G either in the interior or the exterior region. Additionally, they must be discontinuous in θ to circumvent the singularities at $\theta = 0, \pi$. Because the associated Legendre functions satisfy

$$L_\theta P_n^{-m} = -n(n+1)\sin^2\theta P_n^{-m}$$

where the operator

$$L_\theta = \sin\theta\partial_\theta \sin\theta\partial_\theta - m^2$$

homogeneous solutions of the form

$$h_\Phi(r, \theta, \phi) = -iG_a(kr) \cdot \sum_{m=0}^{\infty} \begin{Bmatrix} -\sin m\phi \\ \cos m\phi \end{Bmatrix} \begin{matrix} ika\alpha_m P_0^{-m} & \text{(region I)} \\ (-\tilde{\alpha}_m/ika)\bar{P}_0^{-m} & \text{(region II)} \end{matrix}$$

$$h_\Psi(r, \theta, \phi) = +iY_0 G_a(kr) \cdot \sum_{m=0}^{\infty} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \begin{matrix} \beta_m P_0^{-m} & \text{(region I)} \\ \tilde{\beta}_m \bar{P}_0^{-m} & \text{(region II)} \end{matrix}$$

are allowed if

$$\begin{aligned} \beta_m &= ika\,\alpha_m \\ \tilde{\alpha}_m &= ika\,\tilde{\beta}_m \end{aligned} \tag{6}$$

As shown in Figure 1, region II denotes the cone $\{(r, \theta, \phi) \mid \theta \in [\theta_0, \pi]\}$ and region I denotes its complement. The function $G_a(kr) = e^{ik(r-a)}/kr$. For simplicity we have chosen $G_a(kr)$ to be the fundamental solution associated with the operator $\{\partial_r^2 + k^2\}$ times a fixed phase factor so that it equals a convenient real constant, $1/ka$, at $r = a$. The dual associated Legendre functions

$$\bar{P}_n^{-m}(\cos\theta) = (-1)^{n+m} P_n^{-m}(\cos(\pi - \theta))$$

In particular,

$$P_0^{-m}(\cos\theta) = \frac{(-1)^m}{m!}\tan^m\frac{\theta}{2}$$

Note that our sign convention differs from the one used by *Gradshteyn and Ryzhik* [1965, p. 1008] by a factor of $(-1)^m$.

The constraint conditions (6) ensure that the potentials h_Φ and h_Ψ satisfy the open sphere boundary condition equations and that the combined fields they generate are null. This allows considerable flexibility in defining the auxiliary potentials Φ^G and Ψ^G. To be in close analogy with the closed sphere case, we set

$$\Phi^G = h_\Phi \qquad \Psi^G = h_\Psi$$

over the exterior region $(r > a)$ and

$$\Phi^G = \Psi^G = 0$$

over the interior region $(r < a)$.

Of special interest is the fact that the constraint condition (6) ensures that no field contributions arise from the combination of the potentials Φ^G and Ψ^G. This means that $\Phi^G\mathbf{r}$ and $\Psi^G\mathbf{r}$ are actually (discontinuous) gauge potentials and that their inclusion in the analysis simply represents a gauge transformation. Physically, they represent a discontinuous, nonradiating distribution of radial electric and magnetic dipoles that compensate for the discontinuity introduced when the circular hole was cut in the spherical shell.

Notice that we could equally as well have chosen to define Φ^G and Ψ^G nonzero only over the interior region even though it contains the location ($r = 0$) of their sources. This is again due to the fact that the combination of these gauge potentials does not generate any fields. The fields produced by either h_Φ or h_Ψ are singularity-free (divergence-free) because their fields have no radial components and

$$\Delta_\perp h_\Phi = \Delta_\perp h_\Psi = 0$$

where

$$\Delta_\perp = \left(\frac{1}{r^2 \sin^2 \theta}\right)[\sin \theta \partial_\theta \sin \theta \partial_\theta + \partial_\phi^2]$$

The gauge transformation

$$\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}(r, \theta, \phi) \to \begin{pmatrix}\Phi\\ \Psi\end{pmatrix} + \begin{pmatrix}\Phi^G\\ \Psi^G\end{pmatrix}$$

could then be interpreted as one of the third kind in the presence of boundaries according to Nisbet's homogeneous medium definition [*Nisbet*, 1955]; i.e., the actual sources of the fields are not modified by the change in the potentials even though the potentials' sources are. We also could have constructed slight variants of Φ^G and Ψ^G in both regions. Dual series equations identical to the ones given below can be obtained from any of these choices.

The TE_r and TM_r dual series systems can now be "decoupled" with these auxiliary gauge potentials. Explicitly, one obtains *over the metal* a modified version of the closed sphere conditions

$$[r(\Phi_m + \Phi_m^G)](a, \theta) = 0$$

$$\{\partial_r[r(\Psi_m + \Psi_m^G)]\}(a, \theta) = 0$$

and additionally over the aperture

$$\{\partial_r[r(\Phi_m)]\}(a+, \theta) - \{\partial_r[r(\Phi_m)]\}(a-, \theta) = -\{\partial_r[r(\Phi_m^G)]\}(a+, \theta)$$

$$[r(\Psi_m)](a+, \theta) - [r(\Psi_m)](a-, \theta) = -[r(\Psi_m^G)](a+, \theta)$$

and therefore, the TE_r dual series

$$\Phi_m(a, \theta) = \alpha_m P_0^{-m} \qquad \theta \in [0, \theta_0)$$

$$\{\partial_r[r(\Phi_m)]\}(a+, \theta) - \{\partial_r[r(\Phi_m)]\}(a-, \theta) = (-\bar{\alpha}_m/ika)\bar{P}_0^{-m} \qquad \theta \in (\theta_0, \pi]$$

and the TM_r dual series

$$\{\partial_r[r(\Psi_m)]\}(a, \theta) = \beta_m P_0^{-m} \qquad \theta \in [0, \theta_0)$$

$$\Psi_m(a+, \theta) - \Psi_m(a-, \theta) = (\bar{\beta}_m/ika)\bar{P}_0^{-m} \qquad \theta \in (\theta_0, \pi]$$

General TE_r, A_{mn} and α_{mn}, and TM_r, B_{mn} and $\bar{\beta}_{mn}$, solution coefficients that guarantee satisfaction of Meixner's edge conditions are then constructed from these dual series systems [*Ziolkowski and Johnson*, 1987, sect. 4]. Uniqueness of the results is accomplished by recoupling these solutions through the constraint relations (6). The errors of *Radin and Shestopalov* [1974], *Vinogradov et al.* [1981], and *Casey* [1981] resulted from solving these dual series systems but with vanishing right-hand sides; i.e., with the assumption of a complete decoupling ($\alpha_m = \beta_m = \bar{\alpha}_m = \bar{\beta}_m = 0$), which leads to fields which do not satisfy the correct edge conditions.

PERFORATED SCREEN PROBLEM

The solutions of the circular hole/ground-plane problem [*Meixner*, 1948; *Meixner and Andrejewski*, 1950; *Nomura and Katsura*, 1955] can also be reformulated in terms of this pseudodecoupling ansatz. Consider first plane wave scattering from an infinite ground plane ($z = 0$ or $\theta = \pi/2$). The electromagnetic boundary conditions $E_{tan}(z = 0) = 0$ require

$$\begin{aligned}&\cos \phi\{\partial_\rho^2 + k^2\}(\rho\Psi) - \sin \phi\rho^{-1}\{ikY_0\rho^2\partial_z\Phi + \partial_\phi\partial_\rho(\rho\Psi)\} = 0\\ &\sin \phi\{\partial_\rho^2 + k^2\}(\rho\Psi) + \cos \phi\rho^{-1}\{ikY_0\rho^2\partial_z\Phi + \partial_\phi\partial_\rho(\rho\Psi)\} = 0\end{aligned} \tag{7}$$

where $r = \rho$ when $z = 0$. They are satisfied if the potentials comply with the boundary conditions:

$$\begin{aligned}\Psi(\rho, \theta = \pi/2) &= 0\\ \partial_z\Phi(\rho, \theta = \pi/2) &= 0\end{aligned} \tag{8}$$

On the other hand, when there is a circular hole of radius a present centered at the origin, (7) holds only over the metal ($r = \rho > a$, $z = 0$) and must be supplemented by additional equations generated from the boundary conditions; i.e., as in the open sphere case, with H_{tan} continuous over the aperture ($r = \rho < a$,

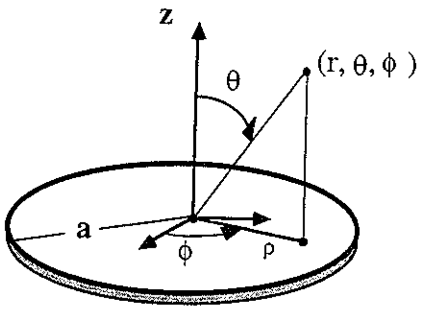

Fig. 2. The scattering of a plane wave from a perfectly conducting circular disc is described by this configuration.

$z = 0$). Equation (8) no longer specifies the solution properly because the TE_r and TM_r modes are now coupled by the hole.

Meixner [1948] constructed a solution of this problem by solving the complementary disc problem and then invoking Babinet's principle. The problem configuration is shown in Figure 2. Because of the geometry, the conditions for the continuity of E_{tan} across $z = 0$ and H_{tan} in the aperture can be satisfied simply by a proper choice of expansion functions. Meixner accomplished this with a transformation of the expansion (3) to one in terms of oblate spheroidal coordinates and their associated wave functions. The unknown expansion coefficients were then determined by satisfying (7) over the disc ($\rho < a$ and $z = 0$). However, if (8) is applied directly as was done by *Möglich* [1927] and *Meixner* [1946], one is led to fields which do not satisfy the correct edge conditions [*Meixner*, 1948].

As above, homogeneous solutions of the boundary condition equations must be introduced. In particular, gauge potentials Φ^G and Ψ^G are constructed from linear combinations of h_Φ, h_Ψ, and their complex conjugates (with $\theta_0 = \pi/2$) in the regions $z > 0$ and $z < 0$ so that over the disc they satisfy (7):

$$\Psi^G\left(\rho, \theta = \frac{\pi}{2}, \phi\right) = -Y_0 \sum_{m=-\infty}^{\infty} [\tilde{\alpha}_m \mathrm{R}\, G_0(k\rho) + \tilde{\beta}_m G_0^*(k\rho)] \cdot \exp(im\phi) \equiv -Y_0\, U(k\rho, \phi)$$

$$\rho\partial_z \Phi^G\left(\rho, \theta = \frac{\pi}{2}, \phi\right) = -\sum_{m=-\infty}^{\infty} \{m[\tilde{\alpha}_m G_0(k\rho) - \tilde{\beta}_m G_0^*(k\rho)]\} \cdot \exp(im\phi) \equiv +\rho V(k\rho, \phi)$$

where $G_0^*(k\rho)$ is the complex conjugate of $G_0(k\rho)$. Notice that these gauge potentials have singularities at $\rho = 0$. The presence of these singularities has caused some confusion as to the validity of Meixner's solution. However, as in the open sphere case, the fields generated by these gauge potentials are divergence-free, hence, singularity-free. This result was noted by *Bouwkamp* [1954, p. 85]: "... the Debye potentials are highly singular at the origin of coordinates. These singularities, however, do not lead to singularities in the field vectors themselves." Meixner's results [*Meixner*, 1948] demonstrate that one can then construct a solution to the disc, hence, to the complementary circular hole problem by requiring that over the disc:

$$\begin{gathered}(\Psi + \Psi^G)(\rho, \theta = \pi/2) = 0 \\ \partial_z(\Phi + \Phi^G)(\rho, \theta = \pi/2) = 0\end{gathered} \tag{9}$$

Uniqueness of the solution is then assured by adjusting all of the $\tilde{\alpha}_m$ and $\tilde{\beta}_m$, hence, the solution coefficients A_{mn} and B_{mn} to produce the proper edge behavior of the fields; i.e., to guarantee satisfaction of Meixner's edge conditions [*Meixner*, 1948; *Jones*, 1964, sect. 9.2]. Therefore, although the solution coefficients A_{mn} and B_{mn} are still coupled through the constants $\tilde{\alpha}_m$ and $\tilde{\beta}_m$, the treatments of the TE_r and TM_r potentials have, in essence, been decoupled as for the closed ground plane problem. Note that Meixner actually treated the potentials by setting

$$\begin{pmatrix}\Phi \\ \Psi\end{pmatrix} = \begin{pmatrix}\Phi^{\text{inc}} \\ \Psi^{\text{inc}}\end{pmatrix} + \begin{pmatrix}\Phi^{1s} \\ \Psi^{1s}\end{pmatrix} + \begin{pmatrix}\Phi^{2s} \\ \Psi^{2s}\end{pmatrix}$$

so that (9) is replaced with

$$\begin{gathered}(\Psi^{\text{inc}} + \Psi^{1s})(\rho, \theta = \pi/2) = 0 \\ \partial_z(\Phi^{\text{inc}} + \Phi^{1s})(\rho, \theta = \pi/2) = 0\end{gathered} \tag{9'}$$

and

$$\begin{gathered}\Psi^{2s}(\rho, \theta = \pi/2) = U(k\rho, \phi) \\ \partial_z \Phi^{2s}(\rho, \theta = \pi/2) = V(k\rho, \phi)\end{gathered} \tag{9''}$$

The computational difficulties associated with Meixner's results [*Meixner*, 1948] actually stimulated the discovery of the much simpler solutions of *Meixner and Andrejewski* [1950] and *Nomura and Katsura* [1955] based upon a Hertzian potential formulation with expansions in terms of cylindrical wavefunctions. It is rather straightforward to show that these results also conform to our "pseudodecoupling" point of view.

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